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DYNAMICS

DYNAMICS

PART II.

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PREFACE

UNTIL recent years the student of that portion of Dynamics which forms the subject matter of Part I. of this volume was unable to proceed to the Dynamics of Rotating Bodies until he had acquired the considerable knowledge of the Calculus necessary for coping with the somewhat formidable treatises on the subject, except by adopting the various 'Calculus dodging' devices which were generally cumbrous and difficult.

The introduction of an elementary treatment of the Calculus into the ordinary School Course of Mathematics now makes it possible for the non-specialist to extend his knowledge of Dynamics as far as the contents of this book, which requires no further acquaintance with the Calculus than the ability to differentiate and integrate x^n. Examples requiring integration of Logarithmic and Trigonometric functions are collected together and their nature indicated; an alternative treatment of Harmonic Motion is provided for those unable to differentiate Trigonometric functions.

Mr. C. H. Ashford, of Dartmouth, has kindly given me

permission to introduce the experiments on pages 239, 286 from his *Elementary Dynamics*, and I am greatly indebted to Dr. W. P. Milne, Dr. S. Brodetsky, and Mr. H. C. Beaven for valuable assistance; also to the Syndics of the Cambridge University Press and the Civil Service Commissioners for permission to use questions from examination papers.

CLIFTON, *September*, 1918.

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PART II.

CHAPTER XI.

DIFFERENTIATION. VELOCITY. PARABOLIC MOTION.

Velocity.

Suppose the distance s feet, travelled in time t secs. by a moving body to be given by the following table :

t	0	1	2	3	4	5
s	1	6	17	34	57	86

Having plotted these values we can obtain the velocity of the moving body at any instant by drawing tangents to the curve and finding the gradient of these tangents as in Pt. I., p. 18.

If, however, the equation of the curve is known, the gradient of the tangent may be obtained by calculation.

Suppose we know the relation between s and t to be an equation of the form $s = pt^2 + qt + r$.

Since when $t=0$, $s=1$, $\therefore\ 1=r$.

Also when $t=1$, $s=6$, $\therefore\ 6=p+q+r$,

and when $t=2$, $s=17$, $\therefore\ 17=4p+2q+r$.

Solving these equations, we find $p=3$, $q=2$, $r=1$.

$\therefore$ the relation between s and t is $s=3t^2+2t+1$.

(By comparing this equation with $s = ut + \frac{1}{2}at^2$, we see that it would represent the motion of a body moving with uniform acceleration $2p$ and starting with velocity q from a point r ft. from the origin, since when $t = 0$, s must equal r.)

Let $s + \delta s$ be the distance travelled in $t + \delta t$ secs.; then

$$s + \delta s = 3(t + \delta t)^2 + 2(t + \delta t) + 1$$
$$= 3t^2 + 6t(\delta t) + 3(\delta t)^2 + 2t + 2(\delta t) + 1,$$

but
$$s = 3t^2 + 2t + 1.$$
$$\therefore \ \delta s = 6t(\delta t) + 3(\delta t)^2 + 2(\delta t).$$
$$\therefore \ \frac{\delta s}{\delta t} = 6t + 2 + 3(\delta t),$$

which gives the average velocity during the interval δt after time t.

If δt is continually diminished, this velocity becomes more and more nearly equal to $6t + 2$, which is defined as being the velocity at time t.

In the language of the Calculus the limit to which $\frac{\delta s}{\delta t}$ continually approaches as δt approaches zero, *i.e.* $\frac{ds}{dt}$ represents the velocity at time t.

When the equation of a curve is given in terms of x and y, $\frac{dy}{dx}$ represents the gradient of the tangent; in like manner when s is given in terms of t, $\frac{ds}{dt}$ also represents the gradient of the tangent, so that we can at once obtain by the rules of differentiation the velocity of a moving body when we know the distance travelled in terms of the time taken.

It is usual to represent differentiation with respect to t by the notation $\dot{s}$ instead of $\frac{ds}{dt}$; similarly, $\frac{dy}{dt}$ is written $\dot{y}$.

If $s = ut + \frac{1}{2}at^2$, we have $\frac{ds}{dt} = u + at$, which is the formula for the velocity when the acceleration is constant.

Acceleration.

When the velocity of a moving body is given in terms of t, we can in a similar manner find the acceleration at any time t.

Suppose the relation to be $v = at^2 + bt + c$. In time $t + \delta t$ let the velocity increase to $v + \delta v$; then

$$v + \delta v = a(t + \delta t)^2 + b(t + \delta t) + c,$$

from which we obtain $\frac{\delta v}{\delta t} = 2at + b + a(\delta t)$.

As δt approaches zero, $\frac{dv}{dt}$ approaches the value $2at + b$, which is the limiting value of $\frac{\delta v}{\delta t}$, and is written $\frac{dv}{dt}$. $\frac{\delta v}{\delta t}$ represents the average acceleration during the interval δt after time t, and the acceleration at time t is the limit of $\frac{\delta v}{\delta t}$, *i.e.* $\frac{dv}{dt}$ or $\dot{v}$.

Since $v = \frac{ds}{dt}$, $\therefore$ acceleration $= \frac{dv}{dt} = \frac{d^2s}{dt^2}$ or $\ddot{s}$.

When $v = u + at$ we have $\frac{dv}{dt} = a$, which shows that the formula refers to motion in which the acceleration is constant.

Angular Motion.

If a body is describing a curve and has travelled a distance s along the arc from A in time t, then in a further interval

of time δt, it will travel a small distance $PQ = \delta s$; $\therefore$ its average velocity during this interval is $\frac{\delta s}{\delta t}$, and its actual

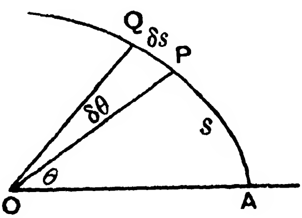

FIG 95

velocity at time t will be $\frac{ds}{dt}$, the limit to which $\frac{\delta s}{\delta t}$ continually approaches as δt approaches zero.

Angular Velocity.

If $A\hat{O}P = \theta$ radians, let $P\hat{O}Q = \delta\theta$, and the radius vector OP then turns through an angle $\delta\theta$ in time δt. $\therefore$ the average angular velocity during the interval δt will be $\frac{\delta\theta}{\delta t}$, and the angular velocity *at* time t will be $\frac{d\theta}{dt}$ or $\dot{\theta}$.

The symbol ω is often used for angular velocity, so that

$$\omega = \frac{d\theta}{dt}.$$

Motion in a Circle.

If the body is moving round the arc of a circle of radius r, then $\delta s = r\,\delta\theta$.

$$\therefore \frac{\delta s}{\delta t} = r\frac{\delta\theta}{\delta t}.$$

As δt approaches zero,

$$\frac{\delta s}{\delta t} \text{ approaches } \frac{ds}{dt} \text{ and } \frac{\delta\theta}{\delta t} \text{ approaches } \frac{d\theta}{dt}.$$

$\therefore \; \dfrac{ds}{dt} = r\dfrac{d\theta}{dt}$, *i.e.* $v = r\omega$, where v is the velocity of the body along the arc at time t.

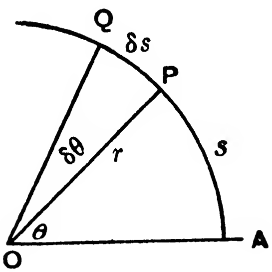

FIG. 96.

Angular Acceleration.

Since acceleration is the rate of change of velocity, the average angular acceleration during the interval δt will be $\dfrac{\delta\omega}{\delta t}$, and the angular acceleration at time t will be

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}.$$

If $\theta = \Omega t + \frac{1}{2}\mathrm{A}t^2$ (cf. $s = ut + \frac{1}{2}at^2$), where Ω is the initial angular velocity, we have $\dfrac{d\theta}{dt} = \Omega + \mathrm{A}t$. Also $\dfrac{d^2\theta}{dt^2} = \mathrm{A}$, which shows that the angular acceleration must have been constant.

EXAMPLE. A bar AC of length 5 ft. has rings at its extremities which pass over two rods AB, BC fixed at right angles. If A moves towards B with a uniform velocity of 1 ft./sec., find the velocity of C, when A is 3 ft. from B.

Let the distances of C and A from B at any instant be x and y respectively, and after a short interval of time δt, let the distances be $x + \delta x$ and $y + \delta y$, where δy will obviously be

negative, if δx is positive. Then $(x+\delta x)^2+(y+\delta y)^2=25$, and $x^2+y^2=25$.

$$\therefore\ x^2+2x(\delta x)+(\delta x)^2+y^2+2y(\delta y)+(\delta y)^2=x^2+y^2.$$

$$\therefore\ 2x(\delta x)+(\delta x)^2+2y(\delta y)+(\delta y)^2=0.$$

Dividing both sides by δt, we have

$$2x\frac{\delta x}{\delta t}+\left(\frac{\delta x}{\delta t}\right)\delta x+2y\left(\frac{\delta y}{\delta t}\right)+\left(\frac{\delta y}{\delta t}\right)\delta y=0.$$

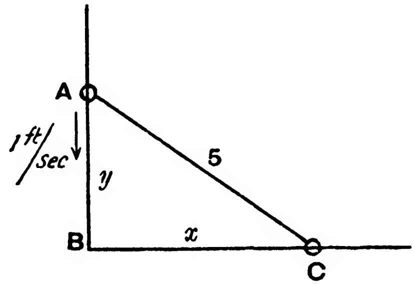

FIG. 97.

As δt approaches 0, δx and δy also approach 0, while $\frac{\delta x}{\delta t}$. $\frac{\delta y}{\delta t}$ become more nearly the velocities $\frac{dx}{dt}$ and $\frac{dy}{dt}$ of the points C and A.

Hence $$2x\frac{dx}{dt}+\left(\frac{dx}{dt}\right)(0)+2y\left(\frac{dy}{dt}\right)+\left(\frac{dy}{dt}\right)0=0,$$

$$i.e.\ \frac{dx}{dt}=-\frac{y}{x}\cdot\frac{dy}{dt}.$$

Now when $y=3$, $x=4$ and $\frac{dy}{dt}=-1$ (negative since y diminishes as t increases),

$$\therefore\ \frac{dx}{dt}=\frac{3}{4}\text{ ft./sec.}$$

EXAMPLES XXVI.

1. The distance s feet travelled by a body in t secs. is given by the following formulae: (1) $s=16t^2$, (2) $s=50t-16t^2$, (3) $s=9t-5$. In each case find the formula giving the velocity

of the body after t secs. Find the velocity in each case at the end of 6 secs.

2. The velocity of a body v ft./sec. acquired in t secs. after starting from a fixed point O is given by the following formulae: (1) $v=32t$, (2) $v=32t+6$, (3) $v=5t^2+2t+4$. In each case find a formula for the acceleration, and find the acceleration when $t=5$.

3. A lamina is rotating round a fixed point O. A line OP in the lamina has rotated through an angle θ from a position OA in time t. If the relation between θ in radians and t in secs. is (1) $\theta=5t+2$, (2) $\theta=4t^2+6t+3$, find in each case the angular velocity and angular acceleration after time t, and their values when $t=2$.

4. Being given the relation $v^2=ks$ connecting the velocity and distance travelled by a body moving in a straight line, deduce an expression for the acceleration.

5. The relation between the velocity v ft./sec. and s ft. is $v=15+\frac{s}{8}$; find the acceleration when $s=480$ ft.

6. If distance travelled in t secs. is represented by

$$s=9t-6t^2+t^3,$$

prove that the velocity is zero when $t=1$ and when $t=3$, and that the acceleration is zero when $t=2$. Describe the motion.

7. A man 6 ft. high walks straight away from a lamp 12 ft. from the ground with uniform velocity of 4 mls./hr. Find the velocity of the end of his shadow.

8. A point moves in a fixed path so that $s=\sqrt{t}$. Prove that the acceleration is negative and that it is proportional to v^3.

9. A ladder 25 ft. long rests with one end on the ground and the other against a vertical wall. When the lower end is 15 ft. from the wall it is pulled away from the wall at the rate of 2 ft./sec.; find the rate at which the upper end descends.

10. A kite moves horizontally at 10 ft./sec. when at a height of 150 ft. from the ground. Find the rate at which the string must be payed out when it is 250 ft. long and supposed to be straight.

11. A man is pulling a boat towards a pier. His hands are 10 ft. higher than the boat, and he is pulling the rope in at 3 ft./sec. How fast is the boat moving through the water when his hands are 20 ft. from the boat?

12. Water is poured into a conical vessel of height 1 ft. and radius of the top 8″, at a uniform rate of 10 cu. in. per sec. Find the volume of water in the vessel when the depth is x'', and find the rate at which the water is rising when the depth is 6″.

13. A ship X sailing due S. at 6 mls./hr. crosses at 4 p.m. the track of a ship Y sailing E. at 8 mls./hr. at a point which Y passed 2 hrs. before. Find at what rate the ships were separating or approaching at 3 p.m. and at 5 p.m., and find at what time the distance between them was not changing.

Projectiles.

If A be the highest point reached by a body projected from O with velocity V when air resistance is ignored, we may find the equation of the curve by taking A as the

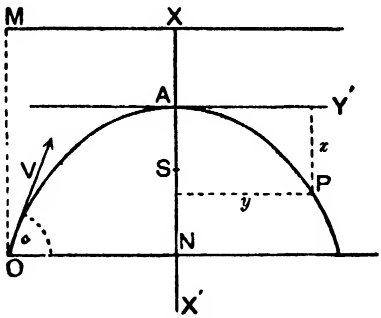

Fig. 98.

origin, and the downward vertical, and the horizontal through A, as axes of x and y respectively.

If the body is at P, t secs. after passing A, we have $y = vt$ (i), where v is the velocity of the body at A, since it moves

horizontally with uniform velocity, there being no acceleration in that direction.

Also $x = \frac{1}{2}gt^2$ (ii), since the body falls a distance x in time t under gravity.

From (i) $t = \frac{y}{v}$.

$$\therefore\ x = \frac{1}{2}g\left(\frac{y}{v}\right)^2. \qquad \therefore\ y^2 = \frac{2v^2}{g}\,.\,x.$$

This is known to be the equation of a parabola, and comparing it with the equation $y^2 = 4ax$, we see that $a = \mathsf{SA} = \frac{v^2}{2g}$, where S is the focus.

$\therefore$ the directrix is at a distance $\frac{v^2}{2g}$ above A and the focus is $\frac{v^2}{2g}$ below A.

If the angle of elevation at O is α, the horizontal component of the velocity is $\mathsf{V}\cos\alpha$, and since this remains unaltered, $v = \mathsf{V}\cos\alpha$.

$$\therefore\ \mathsf{SA} = \frac{v^2}{2g} = \frac{\mathsf{V}^2\cos^2\alpha}{2g}.$$

$\therefore$ the semilatus rectum $2a$ of the parabola is

$$\frac{\mathsf{V}^2\cos^2\alpha}{g}. \quad \text{..........................(iii)}$$

If W is the weight of the body, we have from the energy principle

$$\mathsf{W}\frac{\mathsf{V}^2}{2g} - \mathsf{W}\frac{v^2}{2g} = \mathsf{W}\,.\,\mathsf{AN}.$$

$$\therefore\ \mathsf{V}^2 = v^2 + 2g\mathsf{AN} = 2ga + 2g\mathsf{AN}$$
$$= 2g(\mathsf{AX} + \mathsf{AN}) = 2g\mathsf{MO}, \quad \text{........(iv)}$$

so that the velocity at O equals the velocity the body would have had if it had fallen freely from the directrix.

Since $OM = OS$, $\therefore$ $V^2 = 2g . OS$, so that if V is given, OS is fixed, and the locus of the foci of all paths of a body projected with velocity V from O, is a circle with O as centre and radius $\frac{V^2}{2g}$.

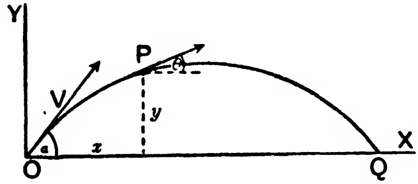

FIG. 99.

If the origin is taken at O, with axes OX horizontal and OY vertical, then the coordinates of P, the position of the projectile after time t, will be given by $x = (V \cos \alpha)t$ and $y = (V \sin \alpha)t - \frac{1}{2}gt^2$.

$$\therefore \; t = \frac{x}{V \cos \alpha}, \qquad \text{(i)}$$

and
$$y = V \sin \alpha . \frac{x}{V \cos \alpha} - \tfrac{1}{2}g . \frac{x^2}{V^2 \cos^2 \alpha}$$

$$= x \tan \alpha - \frac{gx^2}{2V^2 \cos^2 \alpha}. \qquad \text{(ii)}$$

N.B.—It is proved in Coordinate Geometry that the equation of any parabola whose axis is parallel to OY is of the form $y = ax^2 + bx + c$.

The direction of motion of the projectile at time t will be along the tangent to the curve at P, whose slope is given by

$$\frac{dy}{dx} = \tan \alpha - \frac{gx}{V^2 \cos^2 \alpha} \qquad \text{(iii)}$$

$$= \tan \alpha - \frac{g}{V^2 \cos^2 \alpha} . tV \cos \alpha, \text{ from (i)},$$

$$= \tan \alpha - \frac{gt}{V \cos \alpha}.$$

Otherwise. Since at any time t the horizontal velocity is $\frac{dx}{dt}=\mathrm{V}\cos\alpha$, and the vertical velocity $\frac{dy}{dt}=\mathrm{V}\sin\alpha-gt$, the direction of motion is given by

$$\tan\theta=\frac{\mathrm{V}\sin\alpha-gt}{\mathrm{V}\cos\alpha}=\tan\alpha-\frac{gt}{\mathrm{V}\cos\alpha}=\frac{dy}{dx},$$

which illustrates the relation

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}.$$

It will be seen that when $t=0$, $\frac{dy}{dx}$ reduces to $\tan\alpha$, the angle of projection; at the highest point $\frac{dy}{dx}=0$, and from (iii), x then equals

$$\frac{\mathrm{V}^2\sin\alpha\cos\alpha}{g}.$$

$$\therefore \text{ the range } \mathrm{R}=\mathrm{OQ}=\frac{\mathrm{V}^2\sin 2\alpha}{g}. \quad \text{(Fig. 99.)}$$

Range on an Inclined Plane.

A particle is projected with velocity V at an angle α to the horizontal on a plane whose slope is β. If t is the time

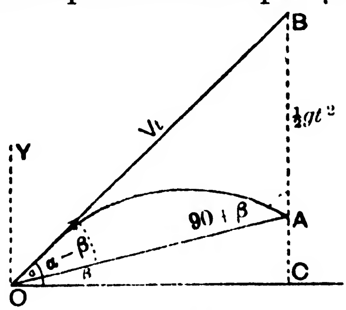

FIG. 100.

of flight and OA the range, then $\mathrm{OB}=\mathrm{V}t$, where BA is vertical. Also $\mathrm{BA}=\frac{1}{2}gt^2$.

Donated by

$\therefore$ from the triangle OBA, since $OBA = 90° - \alpha$,

$$\frac{OA}{\cos\alpha} = \frac{Vt}{\sin(90+\beta)} = \frac{\frac{1}{2}gt^2}{\sin(\alpha-\beta)}.$$

$$\therefore\ t = \frac{2V\sin(\alpha-\beta)}{g\cos\beta} \quad \text{and} \quad OA = \frac{2V^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}.$$

Alternative Method.

The velocity perpendicular to OA is $V\sin(\alpha-\beta)$, and the retardation in this direction is $g\cos\beta$.

Since the distance travelled at right angles to OA is zero, we have $0 = V\sin(\alpha-\beta)t - \frac{1}{2}g\cos\beta t^2$.

$$\therefore\ t = \frac{2V\sin(\alpha-\beta)}{g\cos\beta}.$$

Horizontally the velocity remains $= V\cos\alpha$.

$$\therefore\ OC = V\cos\alpha\,.\,t = \frac{V\cos\alpha\,.\,2V\sin(\alpha-\beta)}{g\cos\beta}$$

and $$OA = OC\sec\beta = \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}.$$

This expression may be written

$$\frac{V^2[\sin(2\alpha-\beta)-\sin\beta]}{g\cos^2\beta},$$

and will be a maximum when $\sin(2\alpha-\beta) = 1$,

i.e. $2\alpha-\beta = 90°$ or $\alpha-\beta = 90°-\alpha$.

$\therefore$ for maximum range the direction of projection must bisect the angle YOA, and the maximum range will be

$$s = \frac{V^2(1-\sin\beta)}{g\cos^2\beta}.$$

EXAMPLE 1. A body is projected with a velocity of 200 ft./sec. at an elevation of 30° from the top of a cliff 300 ft. high. Find how long it takes to reach the sea.

Let P be the point reached in time t if there were no gravitation; then,
$$OP = 200t, \quad PQ = 16t^2.$$
$$\therefore \ PN = 16t^2 - 300.$$
$$\therefore \ \frac{16t^2 - 300}{200t} = \sin 30° = \tfrac{1}{2}.$$
$$\therefore \ 16t^2 - 100t - 300 = 0. \quad \therefore \ t = 8{\cdot}5 \text{ secs.}$$

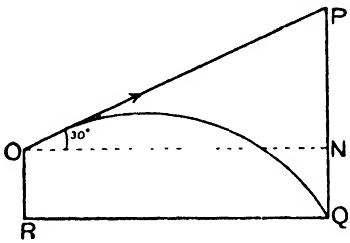

FIG. 101.

Alternative Method.

The vertical component of the initial velocity is
$$200 \sin 30° = 100 \text{ ft./sec.}$$

If we consider the vertical direction measured upwards from O to be positive, then the distance travelled vertically in time $t = -300$ ft.; and since $s = ut + \frac{1}{2}at^2$, we have
$$-300 = 100t - 16t^2.$$
$$\therefore \ 16t^2 - 100t - 300 = 0.$$

EXAMPLE 2. A body is projected at an elevation α to the horizontal from a point on an inclined plane of elevation β. Prove that it hits the plane at right angles if $\cot\beta = 2\tan(\alpha - \beta)$.

If the time of flight is t, the velocity parallel to the plane will then be zero.
$$\therefore \ V\cos(\alpha - \beta) - g\sin\beta t = 0.$$

Also the distance described at right angles to the plane is zero.
$$\therefore \ V\sin(\alpha - \beta)t - \tfrac{1}{2}g\cos\beta t^2 = 0.$$

Hence
$$\cot(\alpha - \beta) = \frac{2\sin\beta}{\cos\beta}.$$
$$\therefore \ \cot\beta = 2\tan(\alpha - \beta).$$

EXAMPLES XXVII.

Assume that air resistance is neglected and $g=32$.

1. A body is projected from the origin O. After t secs. the coordinates of its position P are $x=100t$, $y=80t-16t^2$. Find its horizontal and vertical velocities after t secs.

By eliminating t, find the equation of its path and the slope of the tangent after t secs. Find the velocity and direction of the motion of the particle after 3 secs.

2. The equation of the path of a projectile being $y=x-\dfrac{x^2}{64}$, find the angle at which it was projected and its initial velocity. Find its direction of motion after t secs., and hence find after what time it reaches its highest point and also the coordinates of that point. Find the coordinates of its position when its direction of motion is inclined at 30° to the horizontal.

3. A body projected from O passes through the points (10, 12) and (30, 8). Taking O as origin, find the equation of its path and the angle of projection.

4. A particle is projected from a point A 20 ft. vertically above a point O on the ground. Taking the horizontal and vertical through O as axes, find the equation of the path, given that it goes through the point 60, 30 and hits the ground 100 ft. from O. Find also the direction of motion when reaching the ground.

5. A man in a train which moves with uniform velocity u fires a shot with velocity v at right angles to the direction of the train's motion. Find the greatest range.

6. A hammer thrower takes 8 secs. in getting up the speed of the 16 lb. shot and makes a throw of 180 ft. The gradient of the trajectory at start and finish being estimated at 50°, find the average H.P. at which he works in making the throw.

7. Show that the equation of the trajectory may be written in the form

$$y=x\tan\alpha\left(1-\frac{x}{R}\right),$$

where R is the horizontal range.

8. Use the equation in the form of Qu. 7 to find by how much a bullet misses the mark if the marksman assumes the range to be 300 yds. when it is really 280 yds. with an initial bullet speed of (1) 500 ft./sec., (2) 2000 ft./sec. (This shows the importance of a high velocity bullet.)

9. A jet of water from a hose starts 5 ft. above the ground, just grazes the top of a wall 30 ft. high, distant 50 ft. and strikes the ground at a distance of 70 ft., the distances being taken horizontally from the nozzle. Find the equation of the trajectory, taking the X-axis along the ground and the Y-axis through the nozzle. Find the initial velocity of the jet and its inclination to the horizontal. (C.S.C.)

10. A body is projected at an elevation of 20° to the horizontal with a velocity of 200 ft./sec., and falls on a plane sloping downwards from the point of projection at an angle of 30° to the horizontal. Find the range on this plane.

11. A gun when fired recoils horizontally with velocity U, and the shot leaves the gun with a velocity whose horizontal and vertical components are u and v respectively. Find the angle of elevation of the gun.

'12. A ball is thrown from a point P, 4 ft. above the ground at O. It reaches a maximum height after travelling 30 ft. horizontally, and hits the ground 80 ft. from O. Find the equation of its path referred to horizontal and vertical axes through O.

Parabola of Safety.

When a shell is fired with velocity V from a gun at O the range will depend upon the angle of elevation, and it is a problem of practical importance to find what targets may possibly be reached by this gun.

Let S be the focus of any parabola described by a particle projected with velocity V from O. Then the locus of S is a circle with radius $\frac{V^2}{2g}$ and centre O (see p. 188).

Let $\frac{V^2}{2g} = h = OM$, *i.e.* let the velocity of projection be the velocity due to a fall under gravity from a height h.

Produce OM to M′ so that $MM' = OM = h$, and produce OS to meet the parabola at P.

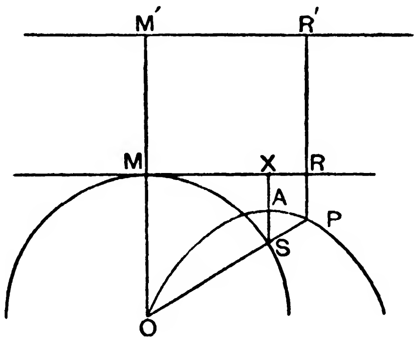

FIG. 102.

Draw PRR′ perpendicular to a line through M′ parallel to MX.

Now, since MXR is the directrix of the parabola OAP, we have $SP = PR$.

$$\therefore \; OP = OS + SP = h + PR = PR'.$$

∴ for different positions of S the locus of P is a parabola with focus O and directrix M′R′.

Let this parabola be MPQ (Fig. 103).

Now if PR is drawn from a point P on a parabola parallel to the axis, it is a well-known property of the parabola that the tangent to the curve bisects the angle SPR, where S is the focus.

But O is the focus of the parabola MPQ and P is a point on that curve. ∴ the bisector of the angle OPR is a tangent to both parabolas at P; hence this enveloping parabola MPQ touches all the parabolas which are the paths of the particles projected from O with velocity V. No particle

projected from O with this velocity can get outside the enveloping parabola whose focus is O, vertex M and directrix M′R′. This parabola is therefore called the parabola of safety.

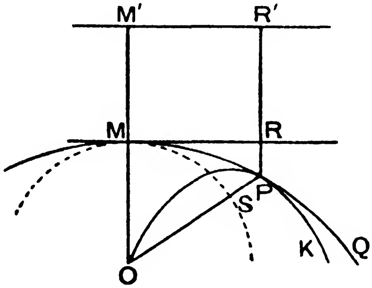

FIG 103.

As the gun is turned to fire in different directions, this parabola will rotate about OM and trace out a paraboloid.

Angle of Elevation.

When a projectile is fired from O with velocity V, where $V^2 = 2gh$, in order to hit a target at P, since the focus S of

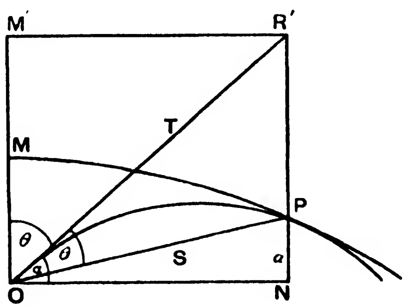

FIG. 104.

the parabolic path lies on OP, OT the tangent to the parabola bisects the angle SOM, where MP is the enveloping parabola.

The greatest range on OP is therefore obtained when the direction of projection bisects the angle SOM (cf. p. 190).

If $TOS=\theta$, $\qquad a=90^\circ-\theta$;

but $O\hat{P}N=2\theta$, $\quad\therefore\ \cos 2\theta=\dfrac{a}{2h-a}$, where $NP=a$,

since $\qquad NR'=2h$ and $OP=PR'$.

If R' is the point on the directrix of the enveloping parabola corresponding to P on the curve, since $PR'=PO$, then $OR'P=\theta$, and therefore the tangent OT at O passes through R'.

EXAMPLES XXVIII.

1. By using the parabola of safety, prove that the greatest range on a plane inclined at an angle β to the horizontal is $\dfrac{V^2}{g(1+\sin\beta)}$.

2. Show also geometrically that the greatest range in Qu. 1 is equal to the distance through which a particle could fall freely during its time of flight.

3. A body is projected from O at an elevation α with a velocity V, where $V^2=2gh$. If the focus of its path is S, prove by projecting OS on the vertical through O that the S.L.R. of the parabolic path is $h(1+\cos 2\alpha)=2h\cos^2\alpha$.

4. The angular elevation of an enemy's position on a hill s ft. above the gun position is β. Show that in order to shell it, the projectile's velocity must be not less than

$$\sqrt{gs\,(1+\operatorname{cosec}\beta)}.$$

5. Show that the maximum range on a horizontal plane of a projectile fired with velocity V, where $V^2=2gh$, from a point P at a height b above the plane is $\sqrt{(b+2h)^2-b^2}$. If K is the point where the projectile hits the plane, prove that the focus of the parabola it describes lies on PK and that the elevation of the gun is α, where $\cos 2\alpha=\dfrac{b}{b+2h}$.

6. Prove that the portion of a vertical wall which can be covered by a jet from a fire engine at a distance c from it is a parabola whose height is $\frac{4h^2-c^2}{4h}$ and breadth is $2\sqrt{4h^2-c^2}$, measured at the level of the nozzle, if the velocity of projection $=\sqrt{2gh}$.

7. Show that a ship S will come under fire of the guns of a fort F which is at a height a ft. above sea level at a range of $2h+a$, but cannot return the fire until the range is reduced to $2h-a$, where $2h$ is the maximum range on the level.

8. Show that the least velocity with which a projectile can be projected from a point A so as to reach a point B is $\sqrt{g(d+h)}$, where d is the distance AB and h the vertical height of B above A.

9. A ship is under fire from the guns of a fort a ft. above sea level. Assuming that the guns of the ship and the fort fire with a velocity V, where $V^2=2gh$, prove that the width of the zone under fire from the fort which the ship has to cross before being able to reply is $2\sqrt{h}\{\sqrt{h+a}-\sqrt{h-a}\}$, and that if $\frac{a}{h}$ is small, this is approximately equal to $2a$.

10. A hose delivers water with a velocity v ft./sec. Find the greatest height above the nozzle the water can reach on a vertical wall x ft. away. If this height is a feet, prove that $\tan(180-2\alpha)=\frac{x}{a}$, where α is the necessary angle of elevation, and show that $\tan\alpha=\frac{v^2}{gx}$.

Circular Motion (variable velocity).

On p. 168 it was stated that when a particle described a circle of radius r with varying velocity the acceleration towards the centre was $\frac{v^2}{r}$. This may be shown as follows.

Let v be the velocity at P, $v+\delta v$ the velocity at Q, where PQ $=\delta s$, the arc travelled in time δt.

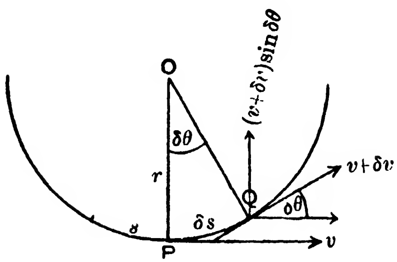

FIG. 105.

The change in velocity in the direction PO in time δt is $(v+\delta v)\sin\delta\theta$, where $P\hat{O}Q=\delta\theta$ radians.

The average acceleration is therefore $\dfrac{(v+\delta v)\sin\delta\theta}{\delta t}$.

Now $\dfrac{\delta s}{\delta t}=v$ (approximately), and will become more nearly equal to v as δt approaches zero.

Also $\delta s=r\,\delta\theta$. $\quad\therefore\ \delta t=\dfrac{r\,\delta\theta}{v}$ approximately.

$$\therefore\ \text{average acceleration}=\frac{(v+\delta v)\sin\delta\theta}{\dfrac{r\,\delta\theta}{v}}\ \text{approximately}$$

$$=\frac{v(v+\delta v)}{r}\cdot\frac{\sin\delta\theta}{\delta\theta}.$$

$\therefore$ acceleration in the direction PO at time t

$$=\underset{\delta\theta\to 0}{\mathrm{Lt}}\ \frac{v(v+\delta v)}{r}\cdot\frac{\sin\delta\theta}{\delta\theta}=\frac{v^2}{r}=\omega^2 r,$$

(where ω is the angular velocity), since $\dfrac{\sin\delta\theta}{\delta\theta}$ approaches unity.

Since the velocity at P along the curve is $\frac{ds}{dt}$, we have $v=\frac{ds}{dt}$, and the acceleration along the tangent at time t is

$$\frac{dv}{dt}=\frac{d^2s}{dt^2}=\ddot{s},$$

which will be zero when v is constant.

Balancing of Rotating Bodies.

When a body at A whose weight is W_1 rotates about an axis at O, the stress P at A along AO necessary to produce the rotation, if we consider the whole mass of the body to be collected at a distance r_1 from O, is given by the equation

$$\frac{P}{W_1}=\frac{\frac{v^2}{r_1}}{g}=\frac{r_1\omega^2}{g},$$

where ω is the angular velocity.

FIG. 106.

There will be a corresponding stress at O in the direction OA $=\frac{W_1r_1\omega^2}{g}$ (the centrifugal force).

This force on the axle increases the pressure on the bearings, produces wear and causes undesirable vibration since it changes its direction as the body rotates.

If another weight W_2 is placed at B a distance r_2 from O in the line AO produced, such that $W_1r_1=W_2r_2$, the total pull at O will be

$$\frac{W_1r_1\omega^2}{g}-\frac{W_2r_2\omega^2}{g},$$

which will be zero, and the weights are then said to be balanced.

Tension in a Revolving Hoop.*

When a hoop is revolving about an axis through its centre at right angles to its plane, the tension produced in the material may be found as follows.

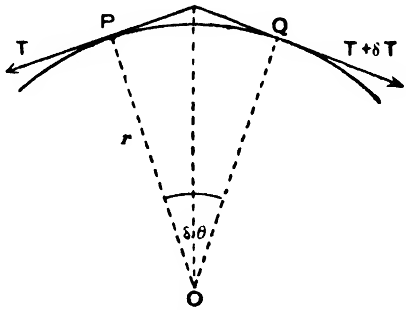

FIG. 107.

Let PQ $= \delta s$ be the length in feet of a small portion of the hoop and $\delta\theta$ (radians) the angle it subtends at the centre.

Let T lbs./sq. ft. be the tension across the section at P and $T + \delta T$ the tension at Q (assumed constant across the section).

If W be the weight of unit volume of the hoop, and A its cross-section in sq. ft., the resultant force on PQ towards O will be

$$A\left[(T+\delta T)\sin\frac{\delta\theta}{2} + T\sin\frac{\delta\theta}{2}\right] = (2T+\delta T)\sin\frac{\delta\theta}{2}\cdot A.$$

(The weight of PQ may be omitted in comparison with these forces.)

$\therefore$ if v is the velocity of PQ,

$$\frac{A(2T+\delta T)\sin\frac{\delta\theta}{2}}{A\,.\,W\,.\,\delta s} = \frac{v^2}{rg},$$

* The rest of the chapter may be omitted on first reading.

i.e.
$$\frac{(2T+\delta T)\sin\frac{\delta\theta}{2}}{Wr\,\delta\theta}=\frac{v^2}{rg},$$

since $\delta s = r\,\delta\theta$,

$$\therefore \frac{(2T+\delta T)}{2Wr}\left(\frac{\sin\frac{\delta\theta}{2}}{\frac{\delta\theta}{2}}\right)=\frac{v^2}{rg}.$$

When $\delta\theta$ is indefinitely diminished, this becomes

$$\frac{T}{Wr}=\frac{v^2}{rg}. \qquad \therefore T=\frac{Wv^2}{g} \text{ lbs./sq. ft.}$$

This result is independent of the radius of the hoop, so that we shall get the same value for T for a flexible belt running over pulleys of any diameter.

EXAMPLE. A tire of cross-section A sq. ft. is shrunk on a wheel of radius r ft. to a tension of T lbs./sq. ft. Find the effect of rotating the wheel with a rim velocity of v ft./sec

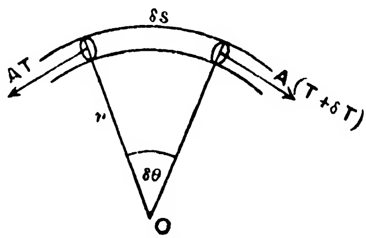

FIG. 108.

When the wheel is at rest let N be the outward pressure of the wheel across the tire per unit length, then the resultant tension on δs towards O is

$$A(T+\delta T)\sin\frac{\delta\theta}{2}+AT\sin\frac{\delta\theta}{2}=A(2T+\delta T)\sin\frac{\delta\theta}{2}.$$

$$\therefore N\,\delta s=A(2T+\delta T)\sin\frac{\delta\theta}{2}.$$

$$\therefore\ \mathsf{N} = \mathsf{AT}\,\frac{d\theta}{ds} \text{ when } \delta s \text{ indefinitely diminishes}$$

$$= \frac{\mathsf{AT}}{r} \text{ lbs. per ft.}$$

The maximum friction couple between the wheel and the tire is $\mu\mathsf{N}(2\pi r)r = \mu 2\pi \mathsf{AT}r$.

When the rim rotates with velocity v, the pressure outwards must be less than the tension inwards, in order to produce a force towards the centre. Let N' be the pressure per unit length outwards on the tire; then

$$\frac{2\mathsf{AT}\sin\frac{\delta\theta}{2} - \mathsf{N}'\delta s}{\mathsf{A}w\,\delta s} = \frac{v^2}{rg},$$

where w is the wt. of 1 cu. ft. of tire.

$$\therefore\ \mathsf{AT}\,\frac{\delta s}{r} - \mathsf{N}'\delta s = \frac{v^2 \mathsf{A}w\,\delta s}{rg}.$$

$$\therefore\ \mathsf{N}' = \frac{\mathsf{A}}{r}\left(\mathsf{T} - \frac{v^2 w}{g}\right),$$

so that the motion causes the normal pressure to diminish, and it appears to be produced by a tension $\mathsf{T} - \frac{v^2 w}{g}$ instead of by T.

The maximum friction couple is now $\mu 2\pi \mathsf{A}\left(\mathsf{T} - \frac{v^2 w}{g}\right)r$, and the tire will become slack when $v = \sqrt{\frac{\mathrm{T}g}{w}}$.

Rotation of the Earth.

If the earth were a sphere, its attraction on a body at its surface would be towards the centre C only if the sphere were at rest. If CN is the axis about which the earth is rotating, a body at A describes a circle whose centre is M. The forces acting on the body are, (i) the earth's pull P along AC, (ii) the pressure of the surface R.

The resultant of these two forces must give a force in the direction AM. The pull of the earth *appears* therefore

to be in the opposite direction to R, and if the body were suspended from a spring balance, the supporting tension would be in the direction of R.

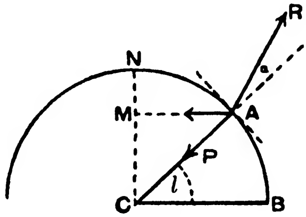

FIG. 109.

If NAB is the meridian passing through A and cutting the equator at B, then $\hat{ACB} = l$ will be the latitude of A.

Let ω be the angular velocity of the earth and r its radius; then the acceleration of the body towards M will be

$$\omega^2 \mathrm{AM} = \omega^2 r \cos l.$$

This may be resolved into

$\omega^2 r \cos^2 l$ along AC

and $\omega^2 r \cos l \sin l$ along the tangent at A.

If the direction of R makes an angle a with CA and g is the acceleration produced on the body when P is the only force acting, we have

$$\frac{\mathrm{P} - \mathrm{R}\cos a}{\mathrm{P}} = \frac{\omega^2 r \cos^2 l}{g} \quad \text{......................(i)}$$

and

$$\frac{\mathrm{R}\sin a}{\mathrm{P}} = \frac{\omega^2 r \cos l \sin l}{g} \quad \text{..................(ii)}$$

Now a is a very small angle, whose cosine is approximately 1, so that from (i) we have

$$\mathrm{R} = \mathrm{P}\left(1 - \frac{\omega^2 r \cos^2 l}{g}\right);$$

hence the rotation of the earth reduces the gravitational pull P upon a body on its surface by

$$\frac{P\omega^2 r \cos^2 l}{g}.$$

At the equator this expression equals less than $\frac{1}{290}$ of the real weight of the body.

EXAMPLES XXIX.

1. A mass of 1 lb. is placed on the spoke of a bicycle wheel 1 ft. from the axle ; find where another mass of 8 oz. should be placed so that when the wheel revolves there may be no resultant centrifugal force.

2. A mass of 15 lbs. is bolted to one of the spokes of a flywheel, its C. of G. being 3 ft. from the centre of the wheel. Another mass of 10 lbs. is bolted to another spoke with its C. of G. 2 ft. from the centre. The angle between the 2 spokes is 120°. Find the resultant force on the bearing of the flywheel due to the inertia of these masses when the wheel is rotating uniformly at 240 revs. a min.

3. The driving wheel of an engine is 7 ft. in diameter, and is loaded with a mass of 150 lbs., which may be considered as concentrated 3 ft. from the centre. Find in tons weight the difference between the greatest and least pressures on the rail due to the presence of this mass when the train is running at 60 mls./hr.

4. A bead of weight W slides down a smooth vertical circle, centre O and radius 2 ft. Find its tangential and normal accelerations when it has fallen a vertical distance of 1 ft. from A, where OA is horizontal. Find also the pressure on the curve at that point.

5. A smooth sphere of radius 1 ft. is fixed on a horizontal plane, and a particle runs down it having been just displaced from rest at the highest point. Find where the particle hits the plane.

6. A rough vertical circle carrying a bead turns in its own plane about its centre with uniform angular velocity greater than $\sqrt{\frac{g}{a}}\left(1+\frac{1}{\mu^2}\right)^{\frac{1}{4}}$, where a is the radius and μ the coefficient of friction. Prove that the bead will never slip.

7. If a flywheel made of steel rotates with rim velocity greater than about 1100 ft./sec. as in a turbine, prove that it must be wire wound to keep it from bursting, given that weight of 1 cu. ft. of steel $=500$ lbs. and that the tension across 1 sq. inch section of steel is 60 tons.

8. Prove that the solid tire of a wheel will become slack when running at more than $\sqrt{\frac{g\pi\, dQ}{W}}$ ft./sec., where W lbs. is the weight of the tire, Q the tension in lbs. wt., and d ft. the diameter of the wheel.

9. A uniform length of wire is bent into a circle and then caused to rotate in its own plane about the centre of the circle. Show that the wire will break when its linear velocity $=\sqrt{gh}$, where h is the maximum length of the wire which can be hung up vertically from one end.

10. A hoop 2″ broad, $\frac{1}{4}$″ thick, is shrunk on to a solid cast-iron wheel 24″ in diameter. The tensile stress in the hoop is 10 tons/sq. in.; if $\mu=0{\cdot}25$, find the force required to draw off the tire.

11. If the weight per cubic foot of the rubber tire of a car is 1 cwt. and the maximum tension is 2 cwt. per sq. in., prove that the maximum safe speed of the car is about 65 mls./hr.

12. Prove that the rotation of the earth, assumed to be a sphere of radius r, causes the direction of the vertical at a place whose latitude is l to make an angle of $\tan^{-1}\left(\frac{\omega^2 r \sin 2l}{2g}\right)$ approximately. with the radius at that place.

13. Show that if the earth were to rotate about 17 times as fast as at present, bodies at the equator would then leave the surface.

14. If the earth were to rotate so fast that bodies at the equator were on the point of leaving the surface, prove that the plumb line at any point will be parallel to the axis of the earth.

CHAPTER XII.

INTEGRATION. DISTANCE INTEGRAL. WORK INTEGRAL.

It has been shown in Pt. I. p. 35, that if PQR is the graph showing the relation between v, the velocity of a moving body, and t the time, then the distance travelled by the body is given by the area bounded by the axes and the curve.

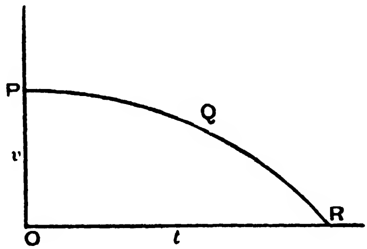

Fig. 110.

If the graph is drawn by plotting corresponding values of v and t as on p. 35, the area is obtained by counting squares, but if the relation between v and t is known, the area may be obtained by calculation.

When y is given in terms of x, the area bounded by the axes and the graph of the given equation is represented in the notation of the Calculus by the definite integral

$$\int_{x_1}^{x_2} y\,dx,$$

where x_1 and x_2 are the limits of the variable x. If then the relation between the velocity of the moving body and the time is given by an equation such as $v=3t^2+2t+5$, the distance travelled by the body in 3 secs. from rest will be

$$\int_0^3 (3t^2+2t+5)\,dt = \Big[t^3+t^2+5t\Big]_0^3 = 51 \text{ ft.}$$

if v is given in ft./sec. and t in secs.

When $v=u+at$ the distance travelled in T secs. from rest will be $\int_0^T (u+at)dt = uT + \frac{aT^2}{2}$.

Velocity Integral.

If the acceleration of a moving body is given in terms of t by a graph such as PQRST, it has been shown in Pt. I. p. 36, that the *increase* of velocity in time OT is given by the area OPQR − area RST.

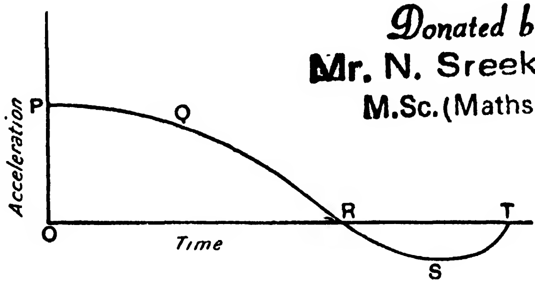

FIG. 111

When the relation between the acceleration and time is given by an equation, the increase of velocity may be calculated by integration in a manner similar to that of the preceding paragraph.

If, for instance, the equation of the curve PQRST is

$a = t^3 - 7t^2 + 7t + 15$, the increase of velocity when $t = 5$ will be

$$\int_0^5 (t^3 - 7t^2 + 7t + 15)\, dt,$$

and if the velocity of the body was zero when $t = 0$, this integral gives the actual velocity of the body.

Part of this area is below the time axis and is therefore negative, but since any area below the axis is negative in the integration, the integral between the limits $t = 0$ and $t = 5$ gives the algebraic sum of the areas OPQR and RST.

If a body starts from rest with constant acceleration a, the velocity after T seconds is given by

$$\int_0^{\mathrm{T}} a\, dt = \Big[at\Big]_0^{\mathrm{T}} = a\mathrm{T},$$

and if the initial velocity is u, the velocity at time T will be $u + a\mathrm{T}$.

If the angular acceleration of a rotating body is constant and equal to A, we have $\frac{d\omega}{dt} = \mathrm{A}$.

$\therefore\ \omega = \mathrm{A}t + k$, and if $\omega = \Omega$ when $t = 0$, we have $k = \Omega$.

$\therefore\ \omega = \Omega + \mathrm{A}t.$

Work Integral.

The method of finding the work done by a variable force has been shown in Pt. I. p. 79.

If the magnitude of the force depends upon the distance x through which it has acted, and if the force P can be expressed algebraically in terms of x, the area enclosed by the axes and the curve which is the graph of the relation between P and x, will be represented by the integral $\int \mathrm{P}\, dx$ which therefore represents the work done by the force.

If the relation is $P = 2x^2 + 3x + 1$, then the work done by P in acting through 3 ft. from $x = 2$ to $x = 5$ is given by

$$\int_2^5 (2x^2 + 3x + 1)\,dx = \left[\frac{2x^3}{3} + \frac{3x^2}{2} + x\right]_2^5$$

$$= \tfrac{2}{3}(5^3 - 2^3) + \tfrac{3}{2}(5^2 - 2^2) + (5 - 2)$$

$= 112\tfrac{1}{2}$ ft.-lbs. if P is in lbs. wt. and x in feet.

Work done by a Couple.

In many cases the forces forming a couple depend upon the angle through which the body has been twisted from its position of equilibrium when no couple acted.

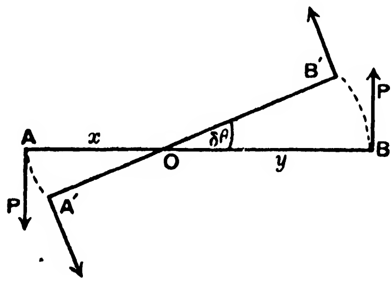

FIG. 112.

Suppose that when the angle of twist is θ radians, the moment of the couple is P(AB).

If we now twist the body through a small angle $\delta\theta$, we may suppose that since $\delta\theta$ is very small the value of P will remain practically unaltered, so that approximately the work done by the forces P in moving through the small arcs AA′ and BB′ will be

P(arc AA′) + P(arc BB′)

$= P\,(OA,\ \delta\theta) + P\,(OB,\ \delta\theta)$

$= P\,(AB)\ \delta\theta = C\ \delta\theta$ if C is the moment of the couple.

If a graph is drawn showing the relation between C, the torque, and θ the angle in radians through which the couple has acted, we can find the work done by the graphical method as before.

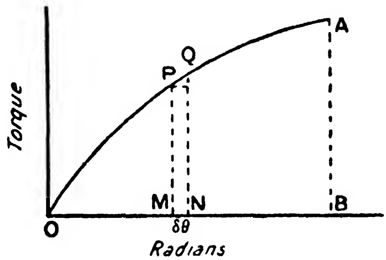

FIG. 113.

Let PM be the magnitude of the torque, for a value OM of θ. If the couple acts through a further small angle $\delta\theta$, the work done is approximately represented by C . $\delta\theta$, the area of the rectangle PN. For a twist through an angle OB, the work done will be represented approximately by the sum of all the rectangles such as PN. As $\delta\theta$ is made smaller, this sum continually approaches the area OPAB, which therefore represents the total work done by the couple.

If the algebraical relation between C and θ is known, this area is calculated from the definite integral $\int_{\theta_1}^{\theta_2} \mathrm{C}\,d\theta$. Suppose $\mathrm{C}=3\theta^2+\theta+2$, then the work done in twisting the body from $\theta=45°$ to $\theta=90°$ is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(3\theta^2+\theta+2)d\theta=\left[\theta^3+\frac{\theta^2}{2}+2\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=3{\cdot}0 \text{ ft.-lbs.},$$

if C is given in lbs.-ft. when θ is in radians.

Work done by a Pulley Belt.

Let T_1 lbs. be the tension on the taut side of the belt and T_2 the tension in the slack side.

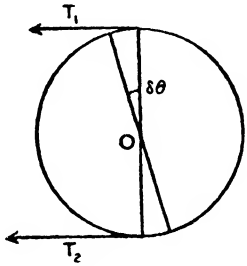

FIG. 114.

Suppose the pulley of radius r to turn through a small angle $\delta\theta$ radians, then the work done to rotate the pulley will be $(T_1 r\,\delta\theta - T_2 r\,\delta\theta) = (T_1 - T_2) r\,\delta\theta$ ft.-lbs.

If the pulley is rotating n times per minute it turns through $\frac{2\pi n}{60}$ radians per sec., and therefore through $\delta\theta$ radians in $\frac{60 \times \delta\theta}{2\pi n}$ secs.

$$\therefore \text{ work done is } (T_1 - T_2) r\,\delta\theta \text{ ft.-lbs. in } \frac{60 \times \delta\theta}{2\pi n} \text{ secs.,}$$

or
$$\frac{(T_1 - T_2) r 2\pi n}{60} \text{ ft.-lbs. per sec.,}$$

and the H.P. transmitted $= \frac{(T_1 - T_2) 2\pi n r}{60 \times 550}$.

Work done in raising Water.

Suppose all the water in the vessel CAED is required to be raised to the level CD.

Imagine the water to be in layers such as AA′B′B, which is at a distance x_1 from the surface.

Let the depth of A′B′ be $x_1+\delta x$ and the weight of water contained in the layer be δW_1.

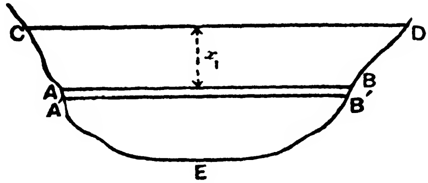

FIG. 115.

The work done to raise this layer will be approximately $\delta W_1 x_1$. and the total work done will be approximately $\Sigma \delta W \, . \, x$.

Now let the depth of the C. of G. of the water be $\bar{x}$; then $\bar{x}=\dfrac{\Sigma\delta W \, . \, x}{\Sigma\delta W}$ approximately. (See *Statics*.)

$$\therefore \quad \Sigma\delta W \, . \, x = \bar{x} \, . \, \Sigma\delta W = \bar{x} W.$$

As δx approaches zero, both of these approximations become more nearly accurate. so the total work done is $\bar{x} \, . \, W$, which is the work required to raise the whole weight of water if it were all supposed to be at the depth of its C. of G.

EXAMPLE. A water cart weighs $\frac{1}{2}$ ton and holds $\frac{1}{2}$ ton of water as well. It is drawn at 4 mls./hr. along a level road and all the water leaks out in 1 hr. Taking $\mu=0{\cdot}05$, find the work done in ft.-tons.

The cart travels x miles in $\dfrac{x}{4}$ hrs. and has then lost $\dfrac{x}{8}$ tons of water; the frictional resistance is then $\cdot 05\left(1-\dfrac{x}{8}\right)$ tons. Since the speed is uniform, the work done in travelling a further distance δx miles is $\cdot 05\left(1-\dfrac{x}{8}\right)\delta x$ ml.-tons. $\therefore$ total work done $=\frac{1}{20}\displaystyle\int_0^4\left(1-\frac{x}{8}\right)dx=792$ ft.-tons.

EXAMPLES XXX.

1. If a wheel is rotating with angular velocity ω when it is given an angular retardation of a rads./sec^2., find through how many radians it will turn before coming to rest.

2. A wheel is set in motion and given an angular acceleration of 5 rads./sec^2.; how many turns will it make before its angular velocity is 20 rads./sec.?

3. A wheel rotates with uniform angular acceleration; if it increases its angular velocity from 10 rads./sec. to 50 rads./sec. in 10 secs., find through how many radians it has turned during that time.

4. A shaft is transmitting a couple equal to 2000 lbs.-ft., and is making 150 revolutions per minute. What H.P. does it transmit?

5. A shaft is driven by a belt which travels at the rate of 30 ft./sec. If the tensions of the belt on two sides of the shaft are 50 and 100 lbs. respectively, find the H.P. transmitted.

6. Find the work done in stretching an elastic string from a length of 18″ to 30″, if the natural length is 8″ and it has a length of 16″ when supporting a weight of 2 lbs. The tension is proportional to the extension.

7. Find the work done in stretching an elastic string from a length of a ft. to b ft., if its natural length is l ft., given that a pull of p lbs. stretches it to a length $2l$.

8. When a couple of torque T lbs.-ft. twists a body through θ radians $T = k\theta$. Find the work done by the couple in twisting the body through a radians.

9. When a rod of length l is fixed at one end and twisted at the other end round its axis through an angle θ radians, the couple required is $k\theta$. Find the work done by a couple in twisting this rod through 90° from 180° to 270°, given that the couple required to keep it twisted at 90° is 8 lbs.-ft.

10. Find the average H.P. required to pump out a dock 600 ft. long, 90 ft. broad and 30 ft. deep in 4 hrs. at a uniform rate if the water is delivered at a level 40 ft. above the bottom

of the dock through a pipe of 1 ft. diameter. The efficiency of the pump is 70 per cent. and account is to be taken of the energy of the water when it leaves the pipe.

11. The force of repulsion exerted on unit charge of electricity by a charge E is $\frac{E}{x^2}$ when x is the distance between them. Find the work done in bringing up the unit charge from an infinite distance to a point distant r from the charge E.

12. If the attractive force in dynes between two magnetic poles is $\frac{m_1 m_2}{x^2}$, where m_1, m_2 represent the pole strengths and x their distance apart in cms., find the work done in ergs in moving a pole of strength 2 from a distance of 10 cms. to 20 cms. from a pole of strength 1.

13. A ship of 1000 tons goes 800 ft. after steam is shut off. If the resistance of the water varies as the square of the distance the ship has to go before coming to rest, find the work done by the water to stop it, given that the initial resistance is 20 lbs. wt. per ton.

14. An iron chain of weight 500 lbs., 20 ft. long, whose sp. gr. is 8·0, hangs in water with the upper end in the surface. Find the work required to lift it vertically so that the lower end is just clear of the water.

15. A cube of cork, sp. gr. 0·25, edge 10 cms., floats in water. Find the work done in pushing it down until just immersed.

16. A body weighing 1 ton is supported by a wire rope which weighs 2 lbs. per foot. Find the work done in lifting the body to the surface from a mine 100 ft. deep.

17. A chain 50 ft. long weighing 10 lbs. per foot length is coiled up at the foot of a rough inclined plane of slope $\frac{1}{10}$, μ being 0·5. Find the work done in pulling the chain slowly up until it is all on the plane.

18. The force of gravitation varies inversely as the square of the distance from the centre of the earth. Find the work done in moving 1 ton from the earth's surface to a height m miles above it, taking the radius as r mls. Inside the earth

the force varies directly as the distance from the centre. Find the work done in raising 1 ton to the surface from a point m miles below.

Function of a Function.

When y is expressed as a function of z and z is expressed as a function of x, we are able to find $\frac{dy}{dx}$ by the important relation

$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}.$$

This is an algebraical expression of the fact that if the rate of increase of z with respect to x is $\frac{dz}{dx}$, and the rate of increase of y with respect to z is $\frac{dy}{dz}$, then the rate of increase of y with respect to x is $\frac{dy}{dz}\times\frac{dz}{dx}$, *e.g.* if A moves twice as fast as B and B moves 3 times as fast as C, then A moves 6 times as fast as C.

EXAMPLE 1. If the acceleration of a moving body is constant and equal to a, we can express this acceleration in terms of v and s.

We are given $\frac{dv}{dt}=a$, but $\frac{dv}{dt}=\frac{dv}{ds}\times\frac{ds}{dt}$, since v will depend upon s and s is a function of t.

$$\therefore\ a=\frac{dv}{ds}\times\frac{ds}{dt}=v\frac{dv}{ds},\quad\text{since } \frac{ds}{dt}=v.$$

Now
$$\frac{d(v)^2}{ds}=\frac{d(v)^2}{dv}\times\frac{dv}{ds}=2v\frac{dv}{ds}.$$

$$\therefore\ a=v\frac{dv}{ds}=\frac{1}{2}\frac{d(v^2)}{ds}.$$

$$\therefore\ 2a=\frac{d(v^2)}{ds}.$$

Integrating with respect to s, we have

$$2as = v^2 + k, \text{ where } k \text{ is some constant.}$$

If $v = u$ when $s = 0$, $0 = u^2 + k$. $\therefore$ $k = -u^2$.

$$\therefore 2as = v^2 - u^2, \text{ where } u \text{ is the initial velocity.}$$

Similarly, if the angular acceleration of a rotating body is constant and equal to A, $\frac{d\omega}{dt} = \text{A}$.

$$\therefore \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \text{A}, \text{ i.e. } \omega\frac{d\omega}{d\theta} = \text{A}.$$

Integrating with respect to θ, we have

$$\frac{\omega^2}{2} = \text{A}\theta + k.$$

If Ω is the initial value of ω, then $\omega^2 = \Omega^2 + 2\text{A}\theta$, which corresponds to the formula $v^2 = u^2 + 2as$.

EXAMPLE 2. Find the formula connecting (i) v and s, (ii) v and t, given that the retardation is proportional to v^3, and that the initial velocity is u.

Since $$\frac{dv}{dt} = -kv^3,$$

$$\therefore v\frac{dv}{ds} = -kv^3. \quad \therefore \frac{dv}{ds} = -kv^2.$$

$$\therefore \frac{ds}{dv} = -\frac{1}{kv^2}. \quad \therefore s = \frac{1}{kv} + c.$$

If $s = 0$ when $v = u$, then $0 = \frac{1}{ku} + c$. $\therefore$ $c = -\frac{1}{ku}$.

$$\therefore s = \frac{1}{k}\left(\frac{1}{v} - \frac{1}{u}\right). \quad \text{..............................(i)}$$

Again, since $\frac{dv}{dt} = -kv^3$,

$$\therefore \frac{dt}{dv} = -\frac{1}{kv^3}. \quad \therefore t = \frac{1}{2kv^2} + c.$$

Also when $t = 0$, $v = u$. $\therefore$ $0 = \frac{1}{2ku^2} + c$.

$$\therefore t = \frac{1}{2k}\left(\frac{1}{v^2} - \frac{1}{u^2}\right). \quad \text{..............................(ii)}$$

Energy Equation.

Suppose the force P lbs. wt., acting upon a body of weight W lbs., to be a function of the distance x ft. through which it has acted (*e.g.* $P = kx$, where k is a constant).

The equation of motion will be $\dfrac{P}{W} = \dfrac{\frac{dv}{dt}}{g}$.

Now,
$$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v\frac{dv}{dx}.$$

$$\therefore \frac{P}{W} = \frac{v\frac{dv}{dx}}{g}.$$

$$\therefore P = W\frac{v\frac{dv}{dx}}{g} = \frac{W}{2g}\frac{d(v^2)}{dx}.$$

(If the student is familiar with the differential notation, he will write
$$P = \frac{W}{g}\cdot v\frac{dv}{dx}. \qquad \therefore P\,dx = W\frac{v\,dv}{g}.)$$

$$\therefore \int P\,dx = W\frac{v^2}{2g} + c.$$

Now the work done by $P = \int P\,dx$, the limits being the initial and final values of x through which P acts.

$$\therefore \text{work done} = \int P\,dx = \frac{W}{2g}\cdot v^2 + c.$$

If the initial value of v is u when $x = 0$,

$$\therefore 0 = \frac{W}{2g}u^2 + c.$$

$$\therefore \text{work done} = \int P\,dx = W\frac{v^2}{2g} - W\frac{u^2}{2g} = \text{change of K.E.}$$

Taking $P = kx$, the total work done from $x=0$ to $x=a$ is

$$\int_0^a kx\,dx = \left[\frac{kx^2}{2}\right]_0^a = \frac{ka^2}{2}.$$

Work and Kinetic Energy.

The energy equation for a body acted upon by forces in different directions may be proved thus :

Let the forces acting on the body of wt. W be P_1, P_2, P_3, ... , which make angles θ_1, θ_2, ... with AO, the direction of its motion at A, and suppose the body to move from A to B, where AB equals s and makes an angle φ with AO.

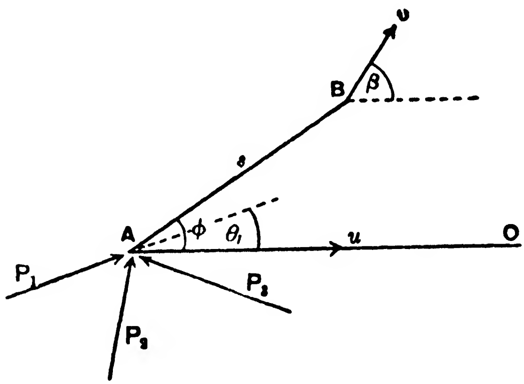

FIG. 116.

Suppose that at A the body is moving with velocity u, and that at B it is moving with velocity v in a direction making an angle β with AO. Then, resolving along AO,

$$[P_1 \cos\theta_1 + P_2 \cos\theta_2 + \dots]\, s \cos\varphi$$
$$= \frac{W}{2g}(v^2 \cos^2\beta - u^2).$$

Perpendicularly to AO,

$$[P_1 \sin\theta_1 + P_2 \sin\theta_2 + \dots]\, s \sin\varphi = \frac{W}{2g}(v^2 \sin^2\beta).$$

By addition,

$$s[P_1(\cos\theta_1 \cos\varphi + \sin\theta_1 \sin\varphi) + P_2 (\qquad) + \ldots] = \frac{W}{2g}(v^2 - u^2).$$

$$\therefore \; s\,[P_1 \cos(\varphi - \theta_1) + P_2 \cos(\varphi - \theta_2) + \ldots] = \frac{W}{2g}(v^2 - u^2).$$

$$P_1 s \cos(\varphi - \theta_1) + P_2 s \cos(\varphi - \theta_2) + \ldots = \frac{W}{2g}(v^2 - u^2),$$

i.e. work done by P_1 + work done by P_2 + ...
= change of K.E. of the body.

Impulse Equation.

When a constant force P acts upon a body of weight W for time t and causes the velocity to change from u to v, we have seen on p. 113 that the equation of motion at once leads to the Impulse equation $Pt = W\frac{v}{g} - W\frac{u}{g}$.

If P is a variable force the impulse produced in time t may be obtained graphically by considering the time divided into small intervals as on p. 79, where the units on OX will now represent time and the impulse will be given by the area under the graph.

This result may be obtained from the equation of motion

$$\frac{P}{W} = \frac{\frac{dv}{dt}}{g}.$$

$\therefore \; P = \frac{W}{g} \cdot \frac{dv}{dt}$, and by integrating with respect to t we have

$$\int P\,dt = W\frac{v}{g} + c.$$

If $v=u$ when $t=0$, $\qquad 0=\mathrm{W}\frac{u}{g}+c.$

$$\therefore \int \mathrm{P}\,dt=\mathrm{W}\frac{v}{g}-\mathrm{W}\frac{u}{g}.$$

If, for instance, a force P lbs. wt. acting upon a body is such that $\mathrm{P}=2t+1$, the impulse given to the body in 2 secs. will be $\int_0^2 (2t+1)\,dt=\Big[t^2+t\Big]_0^2=6$ sec.-lbs.

EXAMPLE. A mass of 1 Kg. on a smooth horizontal table is pulled along the surface by a weight of 50 grams, which just hangs at the edge and is attached to the 1 Kg. by a taut chain 1 metre long, which weighs 10 grams per cm. Find the velocity with which the 1 Kg. reaches the edge.

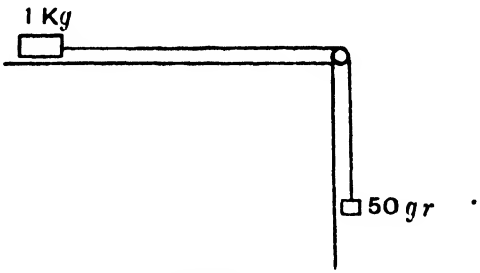

FIG. 117.

The work done on the whole system will be

(i) 50 (100) cm.-grams, since the weight of 50 grams acts through 100 cms.

(ii) 1000 (50) cm.-grams, since the C. of G. of the chain descends a distance of 50 cms. (see p. 213).

Let v be the common velocity; then K.E. gained

$$=(50+1000+1000)\frac{v^2}{2g}.$$

$$\therefore\ 50\,(100)+1000\,(50)=2050\,\frac{v^2}{2\,(981)}.$$

$$\therefore\ v=229{\cdot}5 \text{ cms./sec.}$$

The complete equations of motion, if T is the tension of the chain and x is the distance descended by the 50 grams in time t are:

For the part of the chain hanging vertically

$$\frac{50+10x-\text{T}}{50+10x}=\frac{\frac{dv}{dt}}{g}.$$

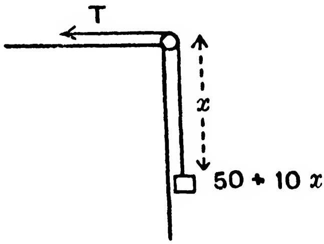

FIG. 118.

For the part of the chain on the table

$$\frac{\text{T}}{1000+10(100-x)}=\frac{\frac{dv}{dt}}{g}.$$

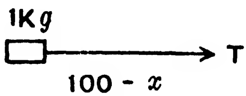

FIG. 119.

$$\therefore\ (2000-10x)\frac{dv}{dt}=g(50+10x)-(50+10x)\frac{dv}{dt}.$$

$$\therefore\ \frac{dv}{dt}=\frac{981(50+10x)}{2050}.$$

$$\therefore\ v\frac{dv}{dx}=\frac{981(50+10x)}{2050}.$$

Integrating with respect to x, we have

$$\frac{v^2}{2}=\frac{981}{2050}(50x+5x^2)+c.$$

When $x=0$, $v=0$. $\therefore$ $c=0$.

$$\therefore \frac{v^2}{2}=\frac{981}{2050}[50(100)+5(100)^2] \text{ when } x=100.$$

$$\therefore v^2=\frac{2(981)}{2050}(55000).$$

$$\therefore v=229\cdot 5 \text{ as before.}$$

EXAMPLES XXXI.

1. A body weighing 50 lbs. is acted upon by a force of $2t+3$ lbs. wt., where t is the time in secs. from rest. Find the velocity of the body, its K.E., and the distance it has travelled when $t=10$.

2. A body weighing 10 lbs. is opposed by a force of $5\sqrt{t}$ lbs. wt. when it is travelling with a velocity of 20 ft./sec. Find how long it takes to come to rest

3. A body weighing 10 lbs. is acted upon by an attracting force of $5x$ lbs. wt., where x is the distance in ft. of the body from a fixed point O in its path. Find its K.E. and its velocity after travelling 5 ft. from a point 10 ft. from O.

4. An electric car starts from rest with an initial acceleration of 2 ft./sec.^{2}, which decreases uniformly with the time. If it takes 35 secs. to acquire its maximum speed, find this speed and the distance then travelled.

5. Two spherical balls attract each other with a force $=km_1m_2r$, where m_1, m_2 are their masses and r the distance apart of their centres. They start when $r=\text{D}$; prove that if C is the sum of their radii, the total K.E. of the balls at impact $=\dfrac{\text{D}^2-\text{C}^2}{2}km_1m_2$.

6. Prove that the energy required to shoot a body weighing W tons from the surface of the earth of radius R vertically upwards to a distance $2a$ from the centre is $\text{WR}\left(1-\dfrac{\text{R}}{2a}\right)$ ft.-tons, given that the force of attraction varies inversely as the square of the distance between the centres.

If $W=10$ tons, $a=3R$ for the distance to the moon, and the nautical mile (1′ of latitude) $=6080$ ft., find the energy required to fire the body to the moon.

7. Find the velocity required to shoot a body to infinity from the surface of the earth.

8. A spiral spring whose tension is proportional to its extension stretches 5″ when a weight of 2 lbs. is attached to its end. It is then pulled down another 5″ and let go; find the velocity of the 2 lbs. as it passes the equilibrium position and also when it is 2″ above that position.

9. Find the extra H.P. that must be exerted by a locomotive to keep up a speed of 45 mls./hr. during the time that it is picking up 2400 gallons of water at a uniform rate from a trough $\frac{1}{4}$ mile long between the rails.

10. The wind resistance area of a racing motor car weighing 2500 lbs. may be taken as 9 sq. ft. and the constant frictional resistance, etc. $=30$ lbs. Taking the wind resistance as due to the impact of the air weighing 0·08 lb./cu. ft. on the above area, find the H.P. necessary for a speed of 90 mls./hr.

11. A fine uniform string of length 9 ft. is in equilibrium passing over a small smooth pulley, and is just displaced; find its velocity when just leaving the pulley.

12. A shot fired at a mark in the horizontal plane through the gun goes a feet beyond it. When a screen of thickness t is placed at the muzzle of the gun perpendicular to the length of the gun, the shot falls b ft. short of the mark. Prove that the shot will hit the mark if the screen is reduced to thickness $\dfrac{at}{a+b}$.

13. A spring required a force of 15 lbs. wt. to stretch it 1″; find what work must be done to increase the extension from 1″ to 2″. This spring is placed horizontally, fixed at one end and connected at the other to a mass of 12 lbs. A shot weighing $\frac{1}{2}$ oz. is fired along the axis of this spring with a velocity of 1500 ft./sec. and embeds itself in the mass. Show that the consequent extension of the spring is $2\frac{1}{8}$″ nearly.

14. A body weighing 5 lbs. lying on a smooth table is acted upon by a horizontal force which varies inversely as the square of the distance of the body from a fixed point O. The body starts from rest 2 ft. from the point; find its velocity when it arrives 1 ft. from O if the initial value of the force is 1 lb. wt.

15. A rope weighing $\frac{1}{2}$ lb. per yard, 6 ft. long, lies on a horizontal table extended at right angles to the edge. When 1 ft. hangs over the edge the rope begins to move. Find its velocity when the other end comes to the edge.

16. A string is stretched between two points A and B, 1 metre apart until its tension is 5 Kg. wt. It is then pulled aside at its mid point in a direction perpendicular to AB through 5 cms. Find the work done by this force, neglecting the extension caused in the string AB.

17. A heavy uniform chain weighing w lbs. per foot is suspended by one end above a horizontal table so that the lower end is just above the table; if it is allowed to fall, show that when x ft. have reached the table the rate at which momentum is being destroyed is $2wx$, and hence the pressure on the table is three times the weight of chain that has reached it.

18. A weight of 100 lbs. hangs freely from the end of a rope. The weight is hauled up from rest by means of a windlass. The pull in the rope starts at 150 lbs. and then diminishes uniformly at the rate of 1 lb. for every foot of rope wound in. Find the velocity of the weight after rising 50 ft.

CHAPTER XIII.

ROTATING BODIES. MOMENTS OF INERTIA.

Kinetic Energy of a Body rotating about a Fixed Axis.

Suppose the figure to represent a section of the body at right angles to the axis through O, about which the body is rotating with angular velocity ω.

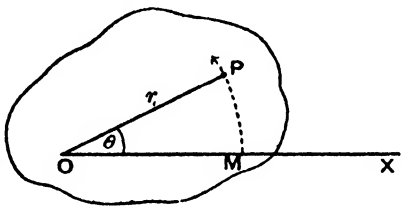

FIG. 120.

It is clear that the portions of the body near O will be moving less rapidly than the particles further away from O, so that the K.E. of the particles depends upon their distance from O.

Let P be the position of a particle of the body of weight δW_1, whose distance from O is r_1. When OP has rotated through an angle θ from its initial position OX, the angular velocity of OP will be $\frac{d\theta}{dt}$, and the angular velocity of the whole body will be $\frac{d\theta}{dt} = \omega$.

The particle P will be moving in a circle round O, and if v is its velocity along the arc MP, then $\frac{v}{r_1} = \omega$.

$\therefore$ the K.E. of the particle at P is

$$\delta W_1 \frac{v^2}{2g} = \delta W_1 \frac{r_1^2 \omega^2}{2g}.$$

Similarly the K.E. of any other particle of weight δW_2 at a distance r_2 from O will be

$$\delta W_2 \frac{r_2^2 \omega^2}{2g}.$$

$\therefore$ the K.E. of the whole body will be

$$\frac{\omega^2}{2g}[\delta W_1 r_1^2 + \delta W_2 r_2^2 + \dots] = \frac{\omega^2}{2g}\Sigma(\delta W) r^2.$$

In order to evaluate this expression we must be able to find $\Sigma(\delta W) r^2$ for the given body, *i.e.* the sum of all terms obtained by multiplying the weight of each particle by the square of its distance from the axis of rotation. The expression $\Sigma(\delta W) r^2$ is called the Moment of Inertia or Second Moment of the body about the given axis.

To find the Moment of Inertia of a thin rod OA, of length $2a$ and weight W about an axis at one end O.

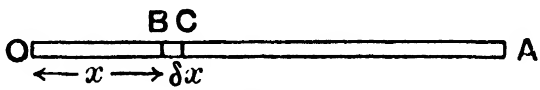

FIG. 121.

Consider a small length δx of the rod at a distance x from O. Let w be the weight of unit length of the rod; then the weight of BC is $w\,\delta x$. The moment of inertia of BC about O lies between $w\,\delta x\,OB^2$ and $w\,\delta x\,OC^2$, and the M.I. of the rod lies between $w\Sigma x^2 \delta x$ and $w\Sigma(x+\delta x)^2 \delta x$.

Suppose that at each point of the rod we erect an ordinate BE(y) such that $y = x^2$; we shall get the curve OED.

The term $x^2\delta x$ is represented by the area of the rectangle BEFC. If we multiplied δx by the distance of the end C from O the term would be represented by the area BE′F′C,

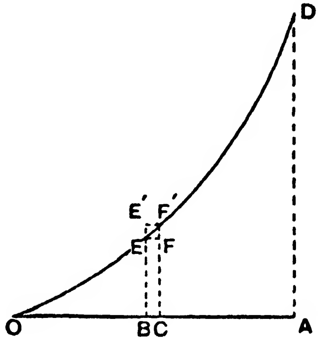

FIG. 122.

since F′C = OC². When this is done for the whole rod we shall get two sets of rectangles, but as δx approaches zero, the sum of both sets of rectangles approaches the area ODA, which therefore gives the expression required. This area is represented by the definite integral $\int_0^{2a} y\,dx = \int_0^{2a} x^2\,dx$, and the Moment of Inertia of the rod is $w\int_0^{2a} x^2\,dx = \frac{8a^3w}{3}$.

But since W is the weight of the whole rod, W $= 2aw$;

$$\therefore \text{ M.I.} = \text{W} \cdot \frac{4a^2}{3}.$$

If the whole material of the rod is supposed collected at a distance $\frac{2a}{\sqrt{3}}$ from O, the M.I. would be

$$\text{W}\left(\frac{2a}{\sqrt{3}}\right)^2 = \text{W}\,\frac{4a^2}{3}.$$

This distance $\frac{2a}{\sqrt{3}}$ is called the radius of gyration or swing radius, and is in general represented by the letter k.

All Moments of Inertia are expressed in terms of the weight of the whole body and the radius of gyration, and they are written $\mathrm{W}k^2$ pounds-(ft.)^2 if W is in pounds and k in feet.

The K.E. of a body rotating with angular velocity ω is therefore $\mathrm{W}k^2\frac{\omega^2}{2g}$ ft.-lbs. if the units are pounds and feet.

Comparing this with the expression for the K.E. of a body weight W moving without rotation with velocity v, we see that $\mathrm{W}k^2$ takes the place of W and ω takes the place of v.

Moment of Inertia of a Rod about its Mid Point.

If w is the weight of unit length, the weight of a small length δx at a distance x from the axis is $w\,\delta x$, and its M.I. about O is $w\,\delta x\,.\,x^2$ approx. If $2a$ is the length of AB, the M.I. of the whole rod is therefore the limiting value of $2w\sum_0^a x^2\,\delta x$.

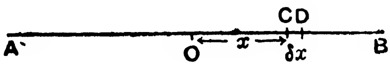

FIG. 123.

As on p. 228, the value which this expression approaches as δx approaches zero is $2w\int_0^a x^2\,dx$.

$$\therefore\ \text{M.I.} = 2w\frac{a^3}{3} = \mathrm{W}\frac{a^2}{3},$$

since $\mathrm{W} = w\,2a$, where W is the weight of the rod.

Moment of Inertia of a circular disc of radius r about an axis through its centre perpendicular to its plane.

In this case we take w to be the weight of unit volume, as all the material inside a ring at a distance x from the centre, of width δx and thickness t, will be at the same distance from the axis through O.

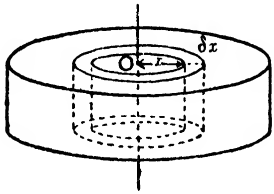

FIG. 124.

The area of this ring at the surface of the disc will be $2\pi x\,\delta x$ approximately, the weight of the ring will be $w\,2\pi x\,\delta x\,t$, and its M.I. $= w\,2\pi x\,\delta x\,t\,.\,x^2$.

$$\therefore \text{ M.I. of disc} = 2\pi wt \sum_0^r x^3\,\delta x \text{ (approx.)}.$$

By drawing ordinates such that $y = x^3$ as on p. 228, we can show that $\sum_0^r x^3\delta x$ approaches the definite integral $\int_0^r x^3\,dx$, as δx approaches zero.

$$\therefore \text{ M.I. of disc} = 2\pi wt\,\frac{r^4}{4} = \text{W}\,\frac{r^2}{2},$$

since W, the weight of the disc, $= \pi r^2 tw$.

It will be noticed that this result does not contain t, and is true for a disc of any thickness.

Theorems on Moments of Inertia.

I. If OX, OY be two axes at right angles to one another in the plane of a *lamina*, and OZ an axis at right angles to

both, then $I_z = I_x + I_y$, where I_x, I_y, I_z are the M.I. about the axes OX, OY, OZ respectively.

Let δw_1 be the weight of a particle at P whose coordinates are $x_1 y_1$; δw_2, the weight of any other particle at $x_2 y_2$, etc.

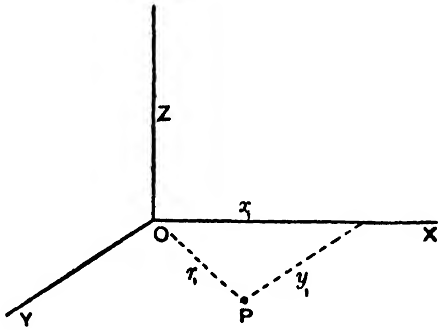

FIG. 125.

Then
$$I_x = (\delta w_1) y_1^2 + (\delta w_2) y_2^2 + \dots,$$
$$I_y = (\delta w_1) x_1^2 + (\delta w_2) x_2^2 + \dots;$$
$$\therefore\ I_x + I_y = \delta w_1 (x_1^2 + y_1^2) + \delta w_2 (x_2^2 + y_2^2) + \dots$$
$$= \delta w_1 r_1^2 + \delta w_2 r_2^2 + \dots,$$

where r_1, r_2, etc., are the distances of the particles from the axis OZ.

$$\therefore\ I_x + I_y = I_z.$$

II. The M.I. of any body about any axis is equal to the M.I. of the body about a parallel axis through the centre of gravity $+ Wa^2$, where a is the distance between the axes.

Let OR be the axis and G the C. of G. of the body. Draw GO from G perpendicular to the axis OR.

Draw GZ parallel to OR and GY perpendicular to both GO and GZ. Take GO, GY, GZ as the three axes of coordinates.

Let the coordinates of a particle at P of weight δw_1 be x_1, y_1, z_1; r_1 its distance from the axis OR, and R_1 its distance from GZ.

M.I. about OR $= \delta w_1 r_1^2 + \delta w_2 r_2^2 + \ldots$

$$= \delta w_1 [(x_1 - a)^2 + y_1^2] + \delta w_2 [(x_2 - a)^2 + y_2^2] + \ldots$$

$$= \delta w_1 (x_1^2 + y_1^2) + \delta w_2 (x_2^2 + y_2^2) + \ldots$$

$$- 2a(\delta w_1 x_1 + \delta w_2 x_2 + \ldots) + a^2 [\delta w_1 + \delta w_2 + \ldots]$$

$$= (\delta w_1 R_1^2 + \delta w_2 R_2^2 + \ldots) + a^2 W$$

$$= I_G + Wa^2.$$

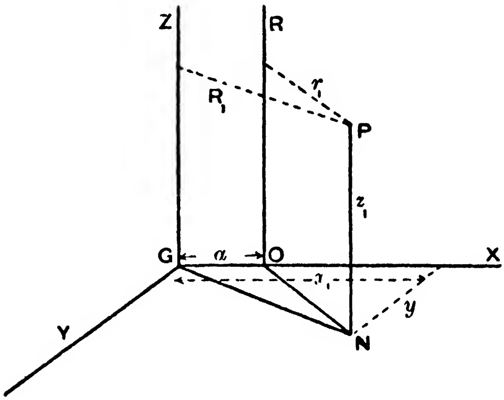

FIG. 126.

N.B.—The coordinates of the C. of G. of a body are given by $\bar{x} = \frac{\Sigma(\delta w)x}{\Sigma \delta w}$, etc., but here $x = 0$, $\therefore$ $\Sigma(\delta w)x = 0$, and the terms $2a(\delta w_1 x_1 + \delta w_2 x_2 + \ldots)$ in the expression for the M.I. disappear.

Moment of Inertia of a Rectangle.

(i) About an axis in its plane bisecting AB and CD.

Let AB, the length of the rectangle, $= 2a$, and AD, the breadth, $= 2b$.

Let w be the weight of unit area and δx the width of a strip at a distance x from EF.

Then M.I. $= 2\int_0^a w\, 2b\, \delta x \,.\, x^2 = \frac{4ba^3 w}{3} = W\frac{a^2}{3}$, since $W = w\, 4ab$.

Similarly the M.I. about GH $= W\frac{b^2}{3}$.

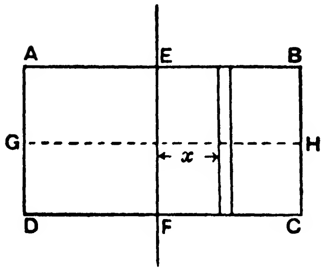

FIG. 127.

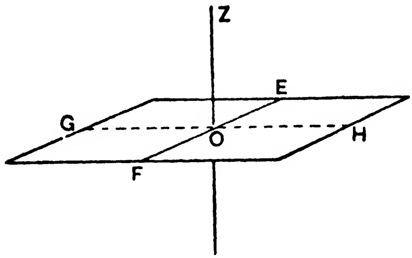

FIG. 128.

(ii) About an axis OZ at right angles to both EF and GH, we have, by Theorem I., since $\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y$,

$$\text{M.I.} = W\frac{(a^2 + b^2)}{3}.$$

Moment of Inertia of a Circle about a Diameter.

We have proved that the M.I. about OZ, an axis through O perpendicular to the plane, is $W\frac{r^2}{2}$ (p. 230).

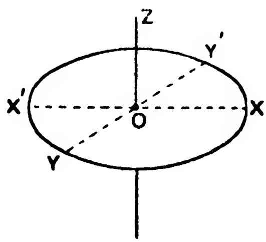

FIG. 129

Now $\mathrm{I}_x = \mathrm{I}_y$, since the M.I. of the disc about XX′ obviously equals the M.I. about YY′.

Since $\mathrm{I}_z = \mathrm{I}_x + \mathrm{I}_y$,

$$\therefore\ W\frac{r^2}{2} = 2\mathrm{I}_x. \qquad \therefore\ \mathrm{I}_x = W\frac{r^2}{4}.$$

Moment of Inertia of a Sphere about a Diameter.

Divide the sphere into sections by planes perpendicular to the X axis. Let PN (y) be the radius of a section of thickness δx at a distance x (ON) from the origin. The moment of inertia about OX of this section, which is approximately a circular disc, is $\mathrm{W}\frac{r^2}{2}$, where $\mathrm{W} = \pi y^2 \,\delta x\, w$ if w is the weight of unit volume and r is $\mathrm{PN} = y$.

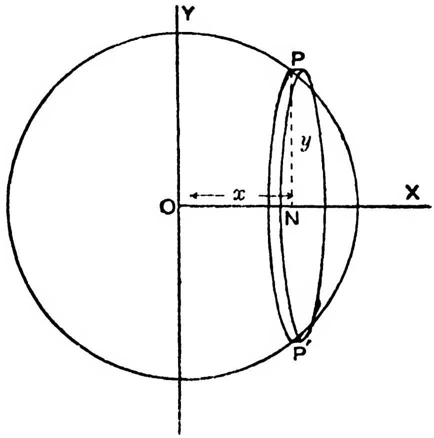

FIG. 130.

∴ M.I. of whole sphere about OX

$$= 2\sum_0^r \frac{\pi y^4\, dx\, w}{2} \text{ approx.}$$

$$= \pi w \int_0^r (r^2 - x^2)^2\, dx \text{ exactly, since } y^2 = r^2 - x^2,$$

$$= \pi w \int_0^r (r^4 - 2r^2x^2 + x^4)\, dx$$

$$= \pi w \left[r^5 - 2\frac{r^5}{3} + \frac{r^5}{5} \right]$$

$$= \pi w \frac{8r^5}{15} = \mathrm{W}\frac{2r^2}{5}, \text{ where W is the wt. of the sphere.}$$

Routh's Rule.

The results we have obtained may easily be remembered by Routh's rule, which states that for any one of three mutually perpendicular axes of symmetry *through the*

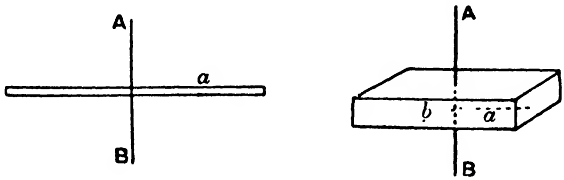

FIG. 131.

C. of G. the M.I. is equal to W × sum of squares of the other two semi-axes divided by 3 for a rectangle, cube or cuboid; 4 for a circle or ellipse and 5 for a sphere or spheroid.

For a thin rod about AB we have $W\dfrac{a^2+0^2}{3}=W\dfrac{a^2}{3}$.

For a cuboid about AB, if the edges at right angles to AB are $2a$ and $2b$, M.I. $= W\dfrac{a^2+b^2}{3}$.

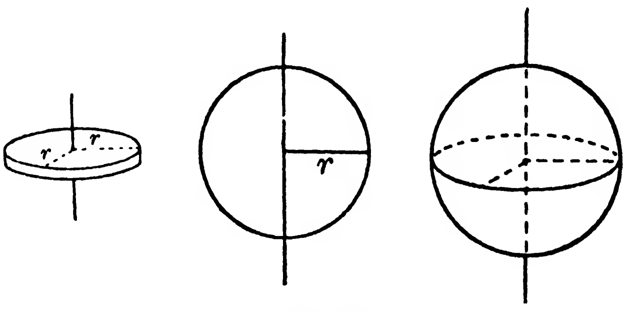

FIG. 132.

For a circular plate of radius r about an axis perpendicular to its plane M.I. $= W\dfrac{(r^2+r^2)}{4} = W\dfrac{r^2}{2}$.

For a circular disc about a diameter

$$\text{M.I.} = \mathrm{W}\,\frac{(r^2+0^2)}{4} = \mathrm{W}\,\frac{r^2}{4}.$$

For a sphere about a diameter

$$\text{M.I.} = \mathrm{W}\,\frac{(r^2+r^2)}{5} = \mathrm{W}\,\frac{2r^2}{5}.$$

EXAMPLE 1. Find the M.I. of a hollow cylinder about its axis, internal radius r_2, external radius r_1, weight W.

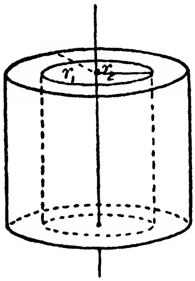

FIG. 133.

The M.I. of the whole cylinder considered as a solid of wt. W_1 is $\mathrm{W}_1 \frac{r_1^2}{2}$. The M.I. of the internal cylinder considered as a solid of the same material is $\mathrm{W}_2 \frac{r_2^2}{2}$.

$$\therefore \ \text{M.I. of the hollow cylinder} = \mathrm{W}_1 \frac{r_1^2}{2} - \mathrm{W}_2 \frac{r_2^2}{2}.$$

Now $$\frac{\mathrm{W}_1}{\mathrm{W}} = \frac{r_1^2}{r_1^2 - r_2^2} \quad \text{and} \quad \frac{\mathrm{W}_2}{\mathrm{W}} = \frac{r_2^2}{r_1^2 - r_2^2}.$$

$$\therefore \ \text{M.I.} = \frac{\mathrm{W}}{2}\,\frac{r_1^4 - r_2^4}{r_1^2 - r_2^2} = \mathrm{W}\,\frac{(r_1^2 + r_2^2)}{2}.$$

EXAMPLE 2. Find the M.I. of a circular disc of radius r and weight W about a tangent.

The M.I. of the disc about a diameter $= \mathrm{W}\,\frac{r^2}{4}$.

$\therefore$ by Theorem II. the M.I. about a tangent

$$= W\frac{r^2}{4} + Wr^2 = W\frac{5r^2}{4}.$$

EXAMPLE 3. Find the M.I. of a solid cylinder, weight W, radius r, about a diameter of one end.

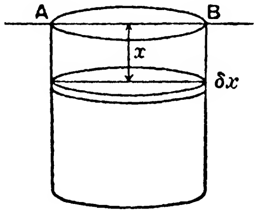

FIG. 134.

Let w be the weight of unit volume. Consider a disc whose distance from the end is x and whose thickness is δx. The M.I. of this disc about a diameter in its plane parallel to AB is $w\pi r^2\,\delta x\,\frac{r^2}{4}$. $\therefore$ its M.I. about the parallel diameter AB is $w\pi r^2\,\delta x\left[\frac{r^2}{4}+x^2\right]$. $\therefore$ the M.I. of the whole cylinder about AB is $w\pi r^2\int_0^l\left(\frac{r^2}{4}+x^2\right)dx$ if l is the length of the cylinder

$$= w\pi r^2\left[\frac{r^2 l}{4}+\frac{l^3}{3}\right] = \frac{w\pi r^2 l}{12}\,[3r^2+4l^2] = \frac{W}{12}\,(3r^2+4l^2)$$

EXAMPLES XXXII.

1. Find the radius of gyration of a thin rod 2 ft. long about an axis at one end.

2. Find the M.I. of a uniform flywheel weighing 50 lbs. whose radius is 1 foot, about an axis through its centre perpendicular to its plane.

3. Find the K.E. of a door weighing 80 lbs. whose breadth is 4 ft., when rotating about its hinges at the rate of 5 radians per sec.

4. Find the K.E. of a uniform flywheel weighing 100 lbs. whose radius is 8 inches, when making 5 revolutions per sec. about its axis.

5. Find the K.E. of a solid cylinder of length 2 ft., radius 6 ins., and weight 20 lbs., when rotating about its axis at the rate of 4 radians per sec.

6. Find the K.E. of a rectangle ABCD, AB = 4 ft., BC = 2 ft., whose weight is 2 lbs., when rotating about an axis perpendicular to its plane through the mid point of AB at the rate of 3 revs. per sec.

7. Find the work which must be done on a uniform flywheel weighing 120 lbs., radius 2 ft., to increase its speed of rotation from 5 to 10 revolutions per sec.

8. Find the M.I. of a flywheel consisting of a uniform ring, internal radius 9″, external radius 12″, weight 20 lbs., about an axis through its centre perpendicular to its plane, the weight of the spokes being negligible.

9. Find the radius of gyration of a thin rod whose length is 4 ft. about an axis 6″ from one end.

10. Find the M.I. of a sphere of radius 1 foot weighing 10 lbs. about a diameter.

11. The inner and outer radii of a hollow cylinder are 2″ and 4″ respectively and its length is 6″. The cylinder weighs 0·25 lb./cu. in. Find its M.I about its axis in lbs.-inch units.

12. Find the M.I. of a circular disc of radius r ft. and weight W lbs. about an axis at right angles to its plane at the extremity of a diameter.

13. A flywheel is 2 ft. in radius and has 3 holes each 1 ft. in diameter cut through it, the centres being symmetrically placed and each 1 ft. from the centre of the axis. Find the radius of gyration of the wheel about an axis through its centre perpendicular to its plane.

14. Find the M.I. of a uniform block of wood weighing 20 lbs., whose length is 10 ft. and cross section a square of edge 1 foot, about a line through its C. of G. parallel to two of the edges of its cross section.

Mass and its Measurement.

When a body supported by a dynamometer is given an acceleration by a sudden motion upwards, the tension registered by the dynamometer is greater than the weight of the body. This additional force is rendered necessary owing to the property of the body known as its Inertia, a term which conveys the idea of reluctance to motion. When we measure the inertia of a body we express it in multiples of a unit called the unit of Mass.

If we wish to determine which of a number of barrels are empty, we can do so by kicking them ; a barrel which moves easily is probably empty, one that moves but little is probably full. The mass of the latter is greater than the mass of the former. It is to be noted that we naturally avoid lifting the barrels, as that would entail the exertion of a force against the pull of the earth, *i.e.* against the weight of the barrels.

The scientific measurement of mass is based on similar considerations. Masses are compared by noting the velocities produced in them by a given force acting for unit time. Assuming that when a spiral spring is stretched to a given length, it will act with a definite force on the body to which it is attached, we find the acceleration, *i.e.* the velocity produced in unit time by this force. If one body A acquires an acceleration twice as great as that acquired by B, the mass of B is said to be twice the mass of A.

In general, masses are inversely proportional to the accelerations produced in them by a given force.

Experiment. Fasten a string to two trolleys with well mounted wheels and pass the string round a pulley (Fig. 135). Draw the pulley horizontally with uniform speed, and note which trolley acquires the greater velocity. Load the other

trolley until both move at the same speed; by definition they will then be of equal mass.

Now weigh the trolleys; it will be found that their weights are the same.

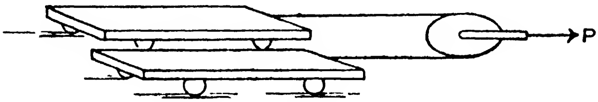

FIG. 135.

In practice it is clear that a method such as the one described above would be unsuitable as a means of determining, for instance, the number of pounds of sugar supplied by a grocer. It now appears that when masses are equal the bodies have the same weight, so that we can compare masses by weighing in a balance.

This result is due to the experimental fact that at a given place the acceleration due to gravity is constant.

Suppose the masses of A and B to be M_1 and M_2, and let their weights be W_1 and W_2.

Let a given force P act upon both, producing an acceleration a_1 in A and a_2 in B.

Then $$\frac{P}{W_1}=\frac{a_1}{g} \quad \text{and} \quad \frac{P}{W_2}=\frac{a_2}{g}.$$

$$\therefore \quad W_1a_1=W_2a_2, \quad i.e. \quad \frac{W_1}{W_2}=\frac{a_2}{a_1}.$$

But by definition $\dfrac{M_1}{M_2}=\dfrac{a_2}{a_1}$. $\quad\therefore\ \dfrac{M_1}{M_2}=\dfrac{W_1}{W_2}$.

Kinetic Energy and Momentum.

It is important to notice that although the expressions $W\dfrac{v^2}{2g}$ and $W\dfrac{v}{g}$ for the Kinetic Energy and Momentum re-

spectively of a moving body involve W, the weight of the body, yet they are clearly dependent only upon the Mass of the body, and will not vary for different parts of the earth's surface. A given bullet fired with a given velocity will penetrate a screen to the same distance on the moon as on the earth. The energy of a rotating flywheel obviously does not depend upon the weight of the wheel which is carried by the axle.

The acceleration g of a given mass falling freely is due to the attraction of the earth, and this force, W, the weight of the body, is by Newton's Second Law proportional to the acceleration it produces. Hence for a given mass $\frac{W}{g}$ is constant. If we take 32·2 lbs. to be the unit of Mass, a body weighing W lbs. at London would contain $\frac{W}{32 \cdot 2}$ such units. This unit is sometimes known as the Engineer's unit of Mass.

For all scientific work the C.G.S. system of units is employed, and the K.E. of a body whose mass is m grams when moving with a velocity of v cms./sec. is $m\frac{v^2}{2}$ ergs. Its momentum is mv seconds-dynes and its M.I. is mk^2 grams-cms^2.

CHAPTER XIV.

LAWS OF ROTATION.

WHEN a string is coiled round the rim of a pulley of radius r which can rotate about an axis through its centre, and the free end of the string is pulled with a force P, the moment of this force about the axis is Pr. The force P produces an additional stress of magnitude P on the bearings, which therefore exert an equal reaction upwards on the axle. The pulley is therefore acted upon by a couple of moment Pr.

If the string unwinds from the pulley, the pulley will continue to rotate until brought to rest by friction at the axle and by air resistance.

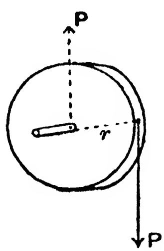

FIG. 136.

The more these forces are reduced the longer will the pulley rotate, and we are hence led to deduce the First Law of Rotation corresponding to Newton's First Law (see Pt. I. p. 47).

First Law of Rotation.

The rate of rotation of a rigid body revolving about an axis fixed in the body and in space, cannot be changed except by the application of an external force which has a moment about the axis, *i.e.* an external torque.

When a body rotates about an axis with uniform angular velocity, we deduce that the resultant couple acting upon it is zero. If, for instance, a pull P just suffices to keep the pulley (Fig. 136) rotating with uniform angular velocity, the moment of the force (Pr) is exactly equal (neglecting air resistance) to the moment of the friction at the axle. This torque is known as the Friction Couple.

Experiment 1. A string is coiled round the axle of a flywheel, and has a weight w attached to one end. w is

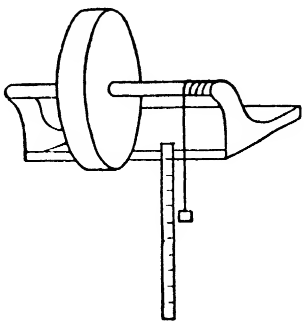

FIG. 137.

allowed to descend in front of a metre scale, and its position noted after 1, 2, 3 ticks of a metronome; or a graph may be drawn on a strip of paper wrapped round the wheel, by a brush carried on a vibrating bar as in the trolley experi-

ment, p. 59. The following results were obtained from the latter method :

t	0	1	2	3	4	5	6	7	8	9	10
s	3	6·35	10	14·05	18·4	23·05	28 05	33·45	39 15	45·15	51·5

By subtracting the readings for the distance fallen, we find the average velocity during each interval of time to be :

t	1	2	3	4	5	6	7	8	9	10
v	3	3·35	3 65	4·05	4·35	4·65	5·0	5·4	5·7	6·0

It appears, therefore, that w descends with constant acceleration, from which it follows that the tension (T) in the string is constant, and hence also the torque Tr acting upon the flywheel, where r is the radius of the axle. If friction is not negligible, the friction couple may be assumed constant.

Now, since w descends with uniform acceleration, the angular acceleration of the flywheel must have been constant. Suppose s to be the length of string uncoiled from the axle, then $\ddot{s}$ is the acceleration of w.

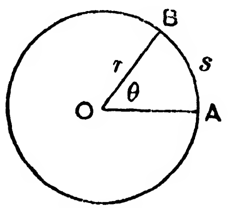

FIG. 138.

But $s = r\theta$. $\quad \therefore \ \ddot{s} = r\ddot{\theta}. \quad \therefore \ \ddot{\theta} = \dfrac{\ddot{s}}{r}.$

Hence, if $\ddot{s}$ is constant, $\ddot{\theta}$ is constant, and we deduce that ;

A constant torque acting upon a rotating body produces a constant angular acceleration.

Experiment 2. Remove the tire from a bicycle wheel and attach masses symmetrically to the spokes so that they can be adjusted at varying distances from the axle. Wind a cord round the rim and attach a weight to the free end.

Let the weight descend when the adjustable masses are close to the rim, and note the time taken by w to fall a given distance.

Then move the masses until they are close to the axle, and again note the time taken by w to fall the same distance as before.

It will at once be noted that w falls more rapidly when the masses are nearer the axle, but the more rapidly w falls the less must be the tension in the string, and consequently the less the torque which is rotating the wheel.

It appears, therefore, that the angular acceleration produced in the wheel is increased by placing the masses nearer the axle even when the torque on the wheel is diminished. With a constant torque the acceleration of the rotating body therefore depends not only on the total mass of the body but on the distribution of its mass relative to the axis; we shall now prove that it depends upon the Moment of Inertia of the body about the axis.

Equation of Motion for a Rigid Body rotating about a Fixed Axis.

Suppose the plane of the paper to cut the fixed axis at O. Let P be any particle of the body in this plane, whose distance from O is r_1, and let its weight be δw_1. Let the angular velocity of the body be $\frac{d\theta}{dt}$. P moves in consequence of

the resultant force produced by the action of all the other particles in contact with it. Let F be this resultant, and suppose it to make an angle φ with OP.

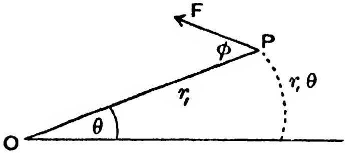

FIG. 130.

Since P is moving round O in a circle, its motion is at right angles to OP, its velocity is $r_1 \frac{d\theta}{dt}$, and its acceleration along its path is $r_1 \frac{d^2\theta}{dt^2}$. But the force acting upon it in this direction is $F \sin \varphi$.

$$\therefore \quad \frac{F \sin \varphi}{\delta w_1} = \frac{r_1 \ddot{\theta}}{g}. \qquad \therefore \; F \sin \varphi = \delta w_1 \frac{r_1 \ddot{\theta}}{g}.$$

$$\therefore \; F \sin \varphi r_1 = \delta w_1 \frac{r_1{}^2 \ddot{\theta}}{g}.$$

Similarly for all the other particles. $\therefore$ for the whole body

$$\Sigma \, F \sin \varphi r = \Sigma \, \delta w \, r^2 \frac{\ddot{\theta}}{g}.$$

Now $\Sigma F \sin \varphi r$ is the sum of the moments of all the forces internal and external acting upon the body. Newton's Third Law, which states that Action and Reaction are equal and opposite, suggests that the internal forces will balance one another, and deductions made on this assumption lead us to believe that such is the case. Hence the terms $\Sigma F \sin \varphi r$ reduce to the sum of the moments of the external forces about O; these will be equivalent to a couple whose moment we will call C.

$$\therefore \; C = Wk^2 \frac{\ddot{\theta}}{g}, \text{ or } \frac{C}{Wk^2} = \frac{\ddot{\theta}}{g}.$$

It will be noted that this equation corresponds to the equation in Pt. I. p. 60 for forces acting on a body which is not rotating, viz. :

$$\frac{P}{W}=\frac{\ddot{x}}{g}.$$

Corresponding to Newton's Second Law, we have therefore the following *Second Law of Rotation* : The angular acceleration produced in a body rotating about an axis fixed in the body and in space is proportional to the value of the external torque.

Angular Momentum Equation.

In linear Dynamics, the equation of motion $\frac{P}{W}=\frac{\ddot{x}}{g}$ leads, by integrating with respect to t, to the Impulse equation

$$\int P\,dt = W\frac{v}{g} - W\frac{u}{g}.$$

In a similar way from the equation for a rotating body

$\frac{C}{Wk^2}=\frac{\frac{d\omega}{dt}}{g}$ we have, assuming C is not a constant,

$$C=\frac{Wk^2}{g}\frac{d\omega}{dt}. \quad \therefore \int C\,dt = Wk^2\frac{\omega}{g}+\lambda.$$

If $\omega=\Omega$ when $t=0$, $\lambda = -Wk^2\frac{\Omega}{g}$.

$\therefore \int C\,dt = Wk^2\frac{\omega}{g} - Wk^2\frac{\Omega}{g}$, which is the equation of Angular Momentum. This equation will be more fully discussed in Chap. XVII.

N.B.—If the torque, and therefore the angular acceleration, is constant, the result may more easily be obtained,

thus: Since $\omega = \Omega + At$, where A is the constant angular acceleration,

$$\therefore A = \frac{\omega - \Omega}{t}.$$

$$\therefore \frac{C}{Wk^2} = \frac{\omega - \Omega}{tg}. \quad \therefore Ct = Wk^2\frac{\omega}{g} - Wk^2\frac{\Omega}{g}.$$

EXAMPLES XXXIII.

1. A wheel weighing 27 lbs., whose radius of gyration is 8 inches, is acted upon by a couple whose moment is 10 lb.-ft. units; find the angular acceleration produced.

2. A wheel weighing 160 lbs., whose radius of gyration is 1 foot, is rotating at the rate of 10 radians per sec. Find the torque necessary to stop it in 1 minute.

3. A cord coiled round the axle, 4 cms. in diameter, of a flywheel maintains a uniform speed of 2 revolutions per sec. when pulled with a force of 20 grams wt. Find the friction couple at the axis.

4. What weight fastened to the cord would be required to maintain the speed of the flywheel in Qu. 3?

5. Find the time taken to make a flywheel of wt. 64 lbs., radius 6″, acquire a speed of 5 revolutions per sec., when a cord coiled round its rim is pulled with a force of $\frac{1}{2}$ lb. wt. (friction being neglected).

6. Find what weight attached to the cord would produce the same effect as the pull of $\frac{1}{2}$ lb. wt. in Qu. 5.

7. A couple of 20 lbs.-ft. is applied to a door weighing 100 lbs., whose width is 4 ft. Find the angular velocity it would produce if it acted for 2 secs. (friction neglected).

8. A wheel weighing 160 lbs., whose radius of gyration is 1 ft. 6 in., is rotating at the rate of 10 revolutions per sec. Find the moment of the friction couple which will stop it in 1 minute, and find the angular retardation it will produce.

9. A uniform rod 1 ft. long, weighing 12 lbs., is rotating about an axis at one end under the action of a constant couple of

moment $\frac{1}{2}$ lb.-ft. What angular acceleration will be produced, and how long will it take to produce an angular velocity of 10 radians per sec. from rest?

10. A circular disc of radius 4 cms., weighing 150 grams, is rotating about a tangent at the rate of 5 turns per sec. Find the frictional couple which will bring it to rest in 10 secs.

Energy Equation for a Body rotating about a Fixed Axis.

Let C be the moment of the couple which is causing rotation; then

$$\frac{C}{Wk^2}=\frac{\ddot{\theta}}{g}, \quad i.e.\ C=Wk^2\frac{\ddot{\theta}}{g}.$$

Now, if ω is the angular velocity of the body at any instant,

$$\omega=\frac{d\theta}{dt}. \qquad \therefore\ \ddot{\theta}=\frac{d\omega}{dt}.$$

But
$$\frac{d\omega}{dt}=\frac{d\omega}{d\theta}\times\frac{d\theta}{dt}=\omega\frac{d\omega}{d\theta}. \qquad \therefore\ \ddot{\theta}=\omega\frac{d\omega}{d\theta}.$$

$$\therefore\ C=\frac{Wk^2}{g}\omega\frac{d\omega}{d\theta}.$$

$$\therefore\ C=\frac{Wk^2}{g}\frac{d}{d\theta}\left(\frac{\omega^2}{2}\right).$$

$$\therefore\int C\,d\theta=Wk^2\frac{\omega^2}{2g}+\lambda.$$

If the initial angular velocity is Ω, when $\omega=\Omega$ the work done by the couple will be zero.

$$\therefore\ 0=Wk^2\frac{\Omega^2}{2g}+\lambda.$$

$$\therefore\int C\,d\theta=Wk^2\frac{\omega^2}{2g}-Wk^2\frac{\Omega^2}{2g},$$

i.e. the work done by the couple equals the change in rotational energy of the body.

N.B.—If the acceleration is constant and equal to A, the proof may be simplified as follows :

Since $\omega^2 = \Omega^2 + 2A\theta, \quad \therefore\ A = \dfrac{\omega^2 - \Omega^2}{2\theta}.$

But $\dfrac{C}{Wk^2} = \dfrac{A}{g}. \quad \therefore\ C = Wk^2\dfrac{\omega^2 - \Omega^2}{2g\theta}.$

$$\therefore\ C\theta = Wk^2\frac{\omega^2}{2g} - Wk^2\frac{\Omega^2}{2g}.$$

Energy Equation for Flywheel and Axle.

If the rotation of the flywheel is produced by the descent of a weight W fastened to a string passing round the axle, let T be the tension of the string ; then the moment of the couple rotating the flywheel and axle is Tr, where r is the radius of the axle.

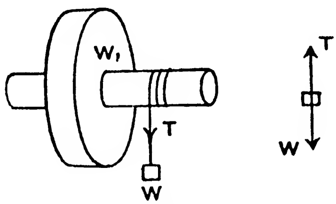

FIG. 140.

(i) $\therefore$ Work done by T = K.E. of flywheel and axle.
Considering the weight W, we have
(ii) Work done by W − work done by T
= K.E. of falling weight.

$\therefore$ by adding (i) and (ii),

Work done by W = K.E. of flywheel and axle
+ K.E. of falling weight.

This result may be deduced from the equations of motion as follows :

For the flywheel and axle, if $W_1 k^2$ is the M.I. :

$$\frac{Tr}{W_1 k^2} = \frac{\ddot{\theta}}{g} = \frac{1}{g}\frac{d\omega}{dt} = \frac{1}{g}\omega\frac{d\omega}{d\theta} = \frac{1}{g}\frac{d}{d\theta}\left(\frac{\omega^2}{2}\right).$$

$$\therefore \quad Tr = \frac{W_1 k^2}{2g}\frac{d(\omega^2)}{d\theta}.$$

Integrating, we have

(i) $$Tr\theta = W_1 k^2 \cdot \frac{\omega^2}{2g},$$

if the system starts from rest.

For the falling weight,

$$\frac{W - T}{W} = \frac{1}{g}\frac{dv}{dt} = \frac{1}{g}v\frac{dv}{dx} = \frac{1}{g}\frac{d}{dx}\left(\frac{v^2}{2}\right),$$

where x is the distance W has fallen.

$\therefore$ by integrating, $$\frac{W - T}{W}x = \frac{v^2}{2g}.$$

(ii) *i.e.* $$Wx - Tx = W\frac{v^2}{2g}.$$

By adding (i) and (ii), since $x = r\theta$, we have

$$Wx = W_1 k^2 \frac{\omega^2}{2g} + W\frac{v^2}{2g}.$$

EXAMPLE 1. A rod of length 1 ft. and weight 5 lbs. can rotate in a vertical plane about an axis at O, one of its ends. If

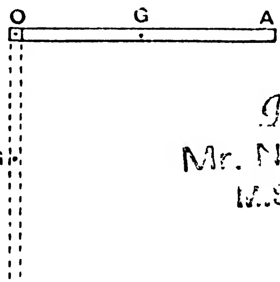

FIG. 141.

it is held horizontally and allowed to drop, find its angular velocity when it reaches the vertical position.

The moment of inertia of the rod about a horizontal axis through G perpendicular to the rod is $W\frac{a^2}{3}$ if $GA=a$. $\therefore$ about a parallel axis through O the

$$\text{M.I.} = W\frac{a^2}{3} + Wa^2 = W\frac{4a^2}{3}.$$

The work done by the weight of the rod in passing to a vertical position is Wa. $\therefore$ if ω be the angular velocity acquired,

$$5 \times 2 = 5 \,.\, \frac{4 \,.\, 2^2}{3}\frac{\omega^2}{2g} \text{ (the K.E. of the rod).}$$

$$\therefore \; \omega = 4{\cdot}9 \text{ rad./sec.}$$

EXAMPLE 2. A pulley of mass 1 lb. and radius 6 ins. has a string passing over it, to the ends of which are fastened weights of 3 lbs. and 5 lbs. Find the velocity of the weights after falling 4 ft. from rest and the pressure on the axle of the pulley during motion if friction is neglected.

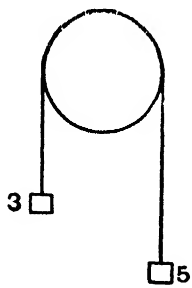

FIG. 142.

The work done on the system to increase K.E. through a distance of 4 ft. is 5×4 ft.-lbs. and to diminish K.E. it is 3×4 ft.-lbs.

If v be the velocity acquired by the weights; $\frac{v}{\frac{1}{2}} = \omega$, the angular velocity of the pulley.

The M.I. of the pulley about the axis is $W\frac{r^2}{2}$.

$$\therefore \; \text{K.E. of the system} = 5\frac{v^2}{2g} + 3\frac{v^2}{2g} + 1\frac{(\frac{1}{2})^2}{2} \,.\, \frac{(2v)^2}{2g},$$

and the work done on the system is (20 – 12) ft.-lbs.

$$\therefore \frac{v^2}{2g}[5+3+\tfrac{1}{2}]=8. \quad \therefore v=7\cdot 76 \text{ ft./sec.}$$

The pulley is set in motion by the tension in the strings; these must therefore be unequal, or there would be no couple acting on the pulley. Let them be T_1 and T_2.

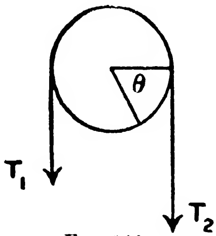

FIG 143.

Consider the forces acting on the pulley. Its equation of motion is

$$\frac{(T_2-T_1)\frac{1}{2}}{\frac{1\cdot(\frac{1}{2})^2}{2}}=\frac{\ddot{\theta}}{g}, \quad \text{.............................(i)}$$

where θ is the angle turned through from rest.

The equations of motion for the weights will be

$$\frac{T_1-3}{3}=\frac{\ddot{x}}{g} \quad \text{and} \quad \frac{5-T_2}{5}=\frac{\ddot{x}}{g},$$

where x is the distance they have moved.

Now, $\qquad \frac{x}{r}=\theta. \quad \therefore \ddot{x}=\frac{1}{2}\ddot{\theta}.$

$$\therefore T_1=3+\frac{3\ddot{x}}{g}. \quad T_2=5-\frac{5\ddot{x}}{g}; \text{ and from (i), } T_2-T_1=\frac{1}{2}\frac{\ddot{x}}{g}.$$

Hence $\qquad T_1=3\tfrac{12}{17}$ lbs. $\quad T_2=3\tfrac{14}{17}$ lbs.

The pressure on the axle during motion equals

$$1+T_1+T_2=8\tfrac{9}{17} \text{ lbs.}$$

EXAMPLES XXXIV.

1. A uniform rod 2 ft. long is hinged at one end so that it can swing in a vertical plane. The rod is held in a horizontal position and then let go. Find its angular velocity when it makes an angle of 60° with the vertical.

2. A door 3 ft. wide weighs 160 lbs. What constant force 2 ft. 6 ins. from the line of hinges and always at right angles to the door will turn it through 90° in 2 secs. ?

3. A top weighs 6 oz. and its radius of gyration is 3 ins. If a string 3 ft. long wound round it is pulled off with a constant force of 1 lb. wt., find the angular velocity the top acquires.

4. A door 8 ft. by 4 ft., weighing 120 lbs., is turned by a constant couple of 20 lbs.-ft. through 90°. Find its angular velocity when it shuts.

5. A trap door 3 ft. wide turns round its hinges from a vertical to a horizontal position. Find the angular velocity acquired if it weighs 30 lbs.

6. A cylindrical drum weighing 10 lbs., radius 1 ft. and radius of gyration 9 ins., can turn without friction about a horizontal axis. A weight of 10 lbs. is fastened to one end of a string coiled round the drum. If the weight is allowed to fall, find the angle through which the drum turns in the first second.

7. A flywheel which can revolve on a horizontal axis weighs 900 lbs., and its radius is r ft. A rope is coiled round its rim and a weight of 90 lbs. hung from its free end, turns the wheel by its descent. At what speed is the weight moving after descending 20 ft. from rest ?

8. A cylindrical barrel, radius 5″, weight 80 lbs., can turn about a horizontal axis about which its swing radius is 4″. A rope is wound on the barrel and supports a bucket of mass 10 lbs. If the bucket descends 20 ft. unchecked to the ground, find what velocity it acquires. If a uniform friction couple of 3 lbs.-ft. resisted the rotation of the barrel, what velocity would the bucket have acquired, and how long will the barrel continue to rotate after the bucket reaches the ground (neglect weight of rope) ?

9. A uniform circular disc weighing 10 lbs., of diameter 9″, is pivoted on a horizontal axis perpendicular to the disc through a point on the circumference. It is displaced through 135° from the position of equilibrium and released. Find its angular velocity and its K.E. when the centre is at the lowest point of its path.

10. A uniform straight rod 6 ft. long swings in a vertical plane about one end; if V is the velocity of the free end of the rod in its lowest position, find the least value of V consistent with the rod making a complete revolution. Compare this with the case of a weight hung by a cord of the same length as the rod and making a complete revolution in a circle.

11. A uniform rectangular door, breadth 3 ft. and mass 50 lbs. has a constant frictional couple at the hinges of 3 lbs.-ft.; find the initial angular velocity that must be given to the door in order that it may swing through a right angle before coming to rest.

12. A weight of 2 lbs. is fastened to a string passing round a horizontal axle of diameter 6″. The axle makes 5 turns in 10 secs., and then the weight reaches the ground. If the axle makes one more turn before coming to rest, find the frictional torque which has been acting (assumed constant throughout).

13. A beam of wood, whose section is a square of 2 feet edge, is 10 feet high and weighs 200 lbs. It stands vertically on a rough floor, and is just displaced so that it turns round one edge of the square. Find its angular velocity on reaching the ground.

14. Two particles of masses 7 m. and 3 m. are fastened to the ends A, B respectively of a weightless rod 15 ft. long, freely hinged to a point O, 5 ft. from A; if the rod is disturbed from its position of unstable equilibrium, find the velocity with which A will pass through its position of stable equilibrium.

15. A string passing over the wheel of an Atwood's machine of radius 4″ has weights of 12 ozs. and 16 ozs. at its ends. The wheel weighs 8 ozs., and its radius of gyration is 3″. Find the velocity of the 16 ozs. after falling 6 ft., assuming a constant frictional torque of $\frac{1}{2}$ oz.-ft. at the axle of the wheel.

16. A pulley on frictionless bearings weighs 10 lbs., and is a disc of uniform thickness. A string passes over it carrying 8 lbs. and 11 lbs. at its extremities. Find the tension in the strings when motion ensues.

17. A flywheel of 3 ft. diameter, weighing 200 lbs., its radius of gyration being 1 ft., is making 800 revs./min. about a horizontal axle, and just winds round its axle a vertical chain weighing 10 lbs./ft. before coming to rest. Find the length of the chain, neglecting its energy.

18. A uniform flywheel weighing 100 lbs., whose diameter is 18″, is mounted on a spindle of diameter 1″, round which is coiled a string supporting a weight. If the speed of the wheel falls from 120 to 100 revolutions per sec. in 30 secs., find the weight which is being wound up if friction and the energy of the spindle are neglected.

MISCELLANEOUS EXAMPLES.

P.

1. A point moves along the line OX so that its distance from O at time t secs. is $t^3 - 12t$ feet. Find its velocity when $t=5$ and its acceleration when $t=2$. At what time will its velocity be zero ?

2. A ballistic pendulum made of a box filled with sand weighing W lbs. and suspended by equal parallel vertical cords of length L ft. is found to recoil through an arc whose chord is l ft. as measured on a tape drawn out in the recoil when struck in a horizontal line through the C. of G. of the box by a shot weighing w lbs. Show that the energy liberated by the impact is $\frac{1}{2}(W + w)\frac{W}{w}\frac{l^2}{L}$ ft.-lbs.

3. A rope AB 20 ft. long is fastened at A and carries 100 lbs. at B. It is held taut horizontally and B is released. When vertical the rope catches against a peg at C, 12 ft. below A. Show that the tension in the rope is doubled, and find whether the mass will describe a complete circle about C as centre.

4. A ball is projected with velocity V from a point on the ground at a distance a from a wall of height b. Prove that it cannot pass over the wall if $V^2 < g\{b+\sqrt{a^2+b^2}\}$, and in this case find the angle of projection that the ball may strike the wall as high as possible.

5. Two equal particles A, B are connected by a taut string, length $2a$, and lie on a rough horizontal plane. B is projected at right angles to the string with such velocity that A remains at rest and B begins to describe an arc of a circle, centre A. Assuming that the string is just clear of the table, prove that a is the length of the greatest arc that B can describe.

Q.

1. A point moves in a straight line so that its distance s from a fixed point O in the line at any time is proportional to t^n. If v be the velocity and f the acceleration at time t, prove that

$$v=\frac{ft}{n-1}, \quad s=\frac{vt}{n}, \quad \frac{1}{s}=\frac{n(n-1)}{ft^2}, \quad v^2=\frac{n}{n-1}fs,$$

and if the average velocity in any interval starting from rest $=\frac{1}{3}$ the final velocity, use these results to find to what power of the distance from the starting point the accleration is proportional.

2. A tug boat A of 400 tons is attached to a ship B of 4000 tons by a cable fixed to A and passing round a capstan on B, the arrangement being such that a pull of 5 tons causes the cable to slip round the capstan. The propulsive force on the tug boat exerted by its propeller is 3 tons, and it starts with the cable slack. At the instant when the cable becomes taut the velocity of A is 5 ft./sec. and B is stationary. Prove that the common velocity of A and B when the cable has just ceased slipping is 1 ft./sec. Find also the time during which the slip occurs and the length of cable passing over the capstan. (Neglect resistance of the water.) (C.U.)

3. A particle moving in a straight line is subject to a resistance which produces the retardation kv^3, where v is the velocity

and k a constant. Show that v and t are given in terms of s by $v=\dfrac{u}{1+ksu}$, $t=\dfrac{s}{u}+\frac{1}{2}ks^2$, where u is the initial velocity.

4. A body weighing 1 ton, starting with a velocity of 10 mls./hr., moves in a straight line, and the power applied is 1 H.P., which remains constant. Find the time that will elapse before the acceleration will be reduced to one-half its initial value. Find also the ratio of the initial acceleration to gravity.

5. The inner and outer radii of a cylindrical flywheel of uniform thickness mounted on a horizontal axis are 6″ and 2 ft. respectively, and its mass is 120 lbs. A string is wrapped round the outer rim and a weight of 80 lbs. hangs from its free end. Find the velocity of this weight when it has fallen 4 ft., neglecting friction and the mass of the spokes.

R.

1. A horizontal arm AB, 4″ long, rotates with uniform angular velocity ω about a vertical axis through A. A light rod BC, 5″ long, is hinged to AB at B, and carries a mass which may be considered as concentrated at C. Find ω in order that BC may become inclined at 30° to the vertical.

2. Assuming the resistance of the air to vary as v^3 and neglecting gravity, prove that $t=as+bs^2$, where t is the time taken by a shot to travel from a gun to a point s ft. away.

3. A spider of weight w hangs suspended by a light elastic thread from the ceiling. The modulus of elasticity is $\dfrac{w}{2}$.* Prove that the work done in climbing to the ceiling is $\frac{1}{3}$ less than it would have been if the spider climbed the same distance on an inelastic thread. (C.U.)

* Experiment shows that the tension per unit area in a stretched wire is proportional to the extension. This is known as Hooke's Law, and is usually written $T=\lambda\dfrac{x-a}{a}$, where $x-a$ is the extension of a string whose original length is a. The constant λ is known as the modulus of elasticity.

4. A weight P lbs. balances a weight of W lbs. on a machine of any kind, in which friction and the weight of the parts of the machine are neglected. Find the force that must be applied to the weight of P lbs. in order to give it an upward or downward acceleration of f ft./sec^2. If, instead, the force is applied to W and imparts the acceleration f to W, prove that this force is to the previous force in the ratio $W^2 : P^2$. (C.S.C.)

5. A small ring fits loosely on a rough spoke of length a of a wheel which can turn about a horizontal axis, and the ring is originally at rest in contact with the lowest part of the rim; if the wheel is now made to revolve with uniform angular velocity ω, prove that the angle through which the wheel will turn before the ring slides is given by the equation

$$\frac{\cos(\theta - \lambda)}{\cos \lambda} + \frac{\omega^2 a}{g} = 0,$$

where λ is the angle of friction.

S.

1. An engine weighing 10 tons starts from rest with a truck of 10 tons, the buffers being originally in contact. When the coupling chain becomes taut, the engine is moving with a velocity of 2 ft./sec. Find the maximum tension in the coupling chain during the ensuing jerk, if each coupling hook is attached to a stiff spring which extends $\frac{3}{4}''$ per ton pull. Assume the chain inextensible.

2. From a gun of mass M lbs., which can recoil freely on a horizontal platform, is fired a shell of mass m lbs., the elevation of the gun being α. Show that the angle ϕ which the path of the shell initially makes with the horizontal is given by the equation $\tan\phi = \left(1 + \frac{m}{M}\right)\tan\alpha$, and assuming that the whole energy of the explosion is transferred to the shell and the gun, show that the muzzle energy of the shot is less than it would have been if the gun were fixed, in the ratio $\frac{M}{M + m\cos^2\phi}$.

3. Given that $t = as + bs^2$, where t is the time in secs. taken to travel s ft. from a point O. If T is the time taken by the

body to travel from A to B, when $AB = L$ ft., prove that its velocity midway between A and B is $= \frac{L}{T}$.

4. A cast iron flywheel, of mean radius 12″ and cross section 4 sq. inches, weighs 450 lbs. per cu. ft. Find the maximum number of revolutions possible per min. if the ultimate tensile strength is 7·5 tons wt./sq. in.

5. A train is running on the level at 30 mls./hr., the frictional resistances being 10 lbs./ton. When steam is shut off and the brakes gradually applied, an additional brake resistance is developed, which increases uniformly from 0 to 40 lbs./ton in 20 secs., and thenceforward the train is subject to a uniform total retarding force of 50 lbs./ton. Find the distance run during the 20 secs. and the velocity at the end of that time. Find also the distance travelled from the instant of commencing to apply the brakes until the train comes to rest.

T.

1. A rod of mass M lbs., length l feet and square section a ft. by a ft. is suspended by a wire attached to the centre of one of the rectangular faces l ft. by a ft. The couple required to hold it when it has been twisted through θ radians from its equilibrium position is $k\theta$ lbs.-ft. If it is let go from this position, find its angular velocity when passing through the equilibrium position.

2. A wall sided barge is 30″ deeper in the water when loaded with 50 tons of coal than when empty. Calculate the area enclosed by the water-line. The interior of the barge is rectangular in shape; the 50 tons of coal fill this space to a depth of 4 ft., and when the barge is thus loaded, the horizontal floor of this space is 3 ft. below water-level. The coal is raised from the barge to a height of 5 ft. above the water-level, the barge gradually rising. Find the expression for the lowering of the coal surface relative to the barge when x tons have been unloaded and the height through which the coal is then being raised, and deduce the work done in raising the whole load. (C.S.C.)

3. A vertical shaft (1 ft. in diameter) turns with one horizontal circular end on a bearing between which and the shaft the coefficient of friction is 0·05. If the vertical pressure on the bearing is 4 tons and the shaft is rotating at 100 revs. per minute, find the H.P. absorbed by friction.

4. Two weights A and B (A>B) are connected by an inelastic string which passes over a smooth pulley. Initially A rests on the ground and B hangs at a height c from the ground. B is then raised through a height h and let fall; show that in order that B may not reach the ground h must be $< \frac{A^2 - B^2}{B^2} \cdot c$, and that if this is the case the system will finally come to rest in time $\frac{A+3B}{A-B}\sqrt{\frac{2h}{g}}$ from the moment B is let fall.

5. As a result of experiments with a rifle it was estimated that the bullet left the muzzle with a velocity of 2400 ft./sec., which was reduced to 2350 ft./sec. after travelling 100 yds. Assuming that air resistance is av^3, and neglecting gravity, calculate the time taken to travel 1000 yards. (C.U.)

Examples requiring a knowledge of differentiation and integration of trigonometric and logarithmic functions.

U.

1. A variable torque produced by a couple of moment M sin θ lbs.-ft. acts on a shaft, M being constant and θ the angle turned through. Find the work done in turning the shaft from $\theta = 0$ to $\theta = \pi$.

2. A projectile should be fired at an angle α to hit a point at a distance R on a horizontal plane. If a small error $\delta\alpha$ is made in α, prove that the difference in the range is

$$2R \cot 2\alpha \,.\, \delta\alpha.$$

3. A body weighing 4 lbs. falls with a velocity of 30 ft./sec. into a liquid which offers a resistance to motion of $\frac{v}{4}$ lbs. wt., where v is the velocity in ft./sec.; find the distance described in the next second.

4. A spherical body is let drop to the ground from a considerable height, the resistance of the air varying as v^2. After a time the body acquires a velocity V and then continues falling with uniform velocity. Show that if v ft./sec. be the velocity at time t the acceleration is given by

$$\frac{dv}{dt}=g\left(1-\frac{v^2}{V^2}\right),\quad \text{and that } t=\frac{V}{2g}\log_e\frac{V+v}{V-v}.$$

If $V=500$ ft./sec., show that the body reaches a velocity of 450 ft./sec. in about 23 secs. ($g=32\cdot2$ ft./sec.). (C.S.C.)

5. A horizontal shaft in fixed bearings carries an eccentric of radius a, the distance of whose centre from that of the shaft is c. The eccentric bears against a horizontal bar which moves vertically parallel to itself between smooth guides. If

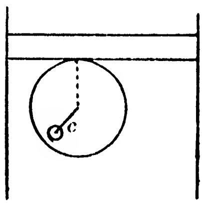

FIG. 144.

μ between the bar and the eccentric $=\frac{1}{4}$ and $a=2c$, find the efficiency of the machine when used for slowly raising the bar from the lowest to the highest position by a half turn of the shaft. (C.U.)

6. A particle is projected from a point on a plane of inclination i at an angle θ with the plane to hit an object on the plane at a distance R. Prove that if θ is small $\theta=\frac{gR\cos i}{2V^2}$, and that if i is small the angle of elevation is therefore the same as if the line of sight were horizontal.

7. The speed of an electric train is reduced in 10 secs. from 30 mls./hr. to 15 mls./hr. by the action of a brake which produces a retardation approximately proportional to v^2. Show that the speed will reduce to 5 mls./hr. in a further 40 secs., and that in this interval the train will travel 483 ft.

8. Oil is flying off a horizontal axle of radius r, rotating with angular velocity ω. Show that no drop reaches a height above the centre line of the axle $> \frac{\omega^2 r^2}{2g} + \frac{g}{2\omega^2}$.

9. In starting a train the pull of the engine on the rails is at first constant and equals P; after the speed attains a certain value u the engine works at a constant rate $R = Pv$. Prove that when the engine has attained a speed $v > u$, the time t and distance x from the start are given by

$$t = \frac{1}{2}\frac{M}{Rg}(v^2 + u^2), \quad x = \frac{1}{3}\frac{M}{Rg}\left(v^3 + \frac{u^3}{2}\right),$$

where M is the mass of the engine and the train.

Calculate the time taken to acquire a speed of 45 mls./hr. when $M = 300$ tons, $R = 420$ H.P., $P = 12$ tons wt. (C.U.)

10. The floor of a level quarry is 60 ft. below the top. A load of 5 cwt. is drawn slowly across the bottom of the quarry by a wire rope from the top. Find the work done in dragging the load 30 ft. if $\mu = 0{\cdot}2$ and if the load is 120 ft. from the side of the quarry when it starts moving.

CHAPTER XV.

HARMONIC MOTION.

Experiment 1. Fasten a weight to the end of a spiral spring which is supported vertically. Pull the weight down and let go. It will be seen to oscillate vertically about the position of equilibrium. Find the average time of an oscillation by timing 5, 10, 20 vibrations. The result will be the same in each case, thus showing that although the range of the vibration is diminishing, yet the time taken for each remains unaltered.

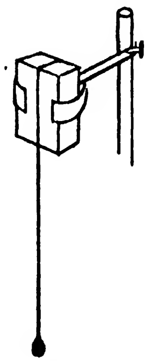

FIG. 145.

Experiment 2. Clamp one end of a thin flat metal bar, say 3 ft. long by 1 inch wide, and draw the free end to one side. When released, it will oscillate about its mean position, and if the time of one vibration is found as in Expt. 1, it will again appear that each vibration takes the same time, if the vibrations are small.

Experiment 3. Fasten a small metal ball to one end of a piece of string, and support the other end by placing it between two small pieces of wood whose lower edges are level and grip these pieces in a clamp. This device keeps the length of string, which is swinging, of constant length when the ball is used as a pendulum.

Again find the time of one oscillation, *i.e.* the interval between two passages of the vertical *in the same direction.* It will again be found that the swings take the same time if they are small. Such vibrations are said to be isochronous (Gr. ἴσος, equal; κρόνος, time).

It can be shown that in each of these cases the force acting on the moving body is proportional to its distance from the position of equilibrium. That this is the case for the spiral spring is shown by the fact that when it is used as a dynamometer the graduations are equidistant, *i.e.* the tension in the spring is proportional to its extension. In fact, most slight displacements which are followed by oscillations about the equilibrium position are due to a restoring force which is proportional to the displacement, *i.e.* $P = kx$, where x is the displacement; such motion is called Harmonic Motion.

When a body of weight W acted upon by such a force is at a distance x from its mean position O, the equation of motion will be

$$\frac{kx}{W} = -\frac{\ddot{x}}{g},$$

the negative sign being due to the fact that the increment in x is measured + to the right, while the force kx acts towards O.

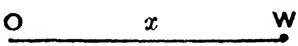

FIG. 146.

Hence $\ddot{x} = -\frac{gk}{W}x$, and the acceleration is therefore also proportional to x.

If the greatest value of x on either side of O is $\pm a$, the acceleration is greatest when $x = a$, *i.e.* at the extreme points of the swing.

Equation of Harmonic Motion.

When the motion is Harmonic, the acceleration is proportional to x and of opposite sign, *i.e.*

$$\ddot{x} = -\mu x \quad \text{or} \quad \ddot{x} + \mu x = 0.$$

This is the differential equation of Harmonic Motion.

* If we take $x = a \cos \sqrt{\mu}\, t$, where a is a constant, it will appear that this value of x satisfies the given equation.

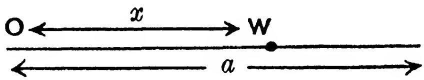

FIG. 147.

Differentiating with respect to t, we have

$$\frac{dx}{dt} = \dot{x} = -\sqrt{\mu}\, a \sin \sqrt{\mu}\, t,$$

and

$$\frac{d^2x}{dt^2} = \ddot{x} = -\mu a \cos \sqrt{\mu}\, t$$

$$= -\mu x, \quad i.e. \ \ddot{x} + \mu x = 0.$$

Therefore the position of the body at any time t is given by $x = a \cos \sqrt{\mu}\, t$.

When $t = 0$, $x = a$, and when $t = \dfrac{\pi}{\sqrt{\mu}}$, $x = -a$. The extreme length of the swing from O is therefore a, and it is called the *Amplitude*.

If we increase t by $\dfrac{2\pi}{\sqrt{\mu}}$ we have

$$x = a \cos \sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right) = a \cos (2\pi + \sqrt{\mu}\, t) = a \cos \sqrt{\mu}\, t,$$

* A direct proof is given on p. 269.

i.e. the body is at the same distance from O after an interval of time $=\dfrac{2\pi}{\sqrt{\mu}}$.

This is called the Periodic time, and as it is independent of a, the time does not depend on the extent or amplitude of the swing.

The velocity at time t is $\dot{x} = -\sqrt{\mu}\, a \sin \sqrt{\mu}\, t$

$$\begin{aligned} &= -\sqrt{\mu}\, a \sqrt{1-\cos^2 \sqrt{\mu}\, t} \\ &= -\sqrt{\mu}\, a \sqrt{1-\frac{x^2}{a^2}} \\ &= -\sqrt{\mu}\, \sqrt{a^2-x^2}. \end{aligned}$$

This will be greatest when $x=0$, and it is zero when $x=\pm a$.

Geometrical Construction for Harmonic Motion.

It has been shown that the distance of a point from its mean position when moving Harmonically is given by $x=a\cos\sqrt{\mu}\, t$. The following method may be used by

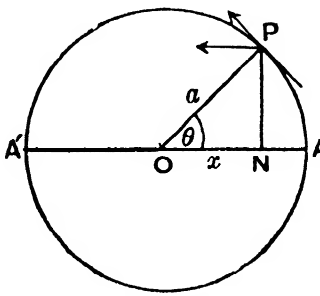

FIG. 148.

those who are not familiar with the derivatives of the trigonometrical functions.

If we suppose a point P to move round a circle of radius a

with uniform angular velocity ω starting from A, in t seconds it will be at P, where $\theta = \omega t$ and N the foot of the perpendicular on OA will be at a distance from O $= a \cos \omega t$. If then $\omega = \sqrt{\mu}$, the position of N will correspond to the position of a body moving harmonically along AA′ under the action of a force which produces an acceleration $= \mu x$. This may be shown independently as follows.

If P is moving with uniform angular velocity ω, its velocity along the curve will be $v = a\omega$ and its acceleration towards O will be $\frac{v^2}{a} = a\omega^2$. The component of this acceleration parallel to NO will be

$$a\omega^2 \cos \theta = a\omega^2 \frac{x}{a} = \omega^2 x.$$

If N keeps pace with P, the acceleration of N will be $\omega^2 x = \mu x$ towards O $\therefore$ N moves harmonically.

The velocity of P resolved parallel to NO will be

$$a\omega \sin \theta = a\omega \sqrt{\frac{a^2 - x^2}{a^2}} = \omega \sqrt{a^2 - x^2}.$$

The periodic time will be the time taken by P to make a complete revolution of the circle. Since it describes ω radians per sec., the time taken to describe 2π radians will be

$$\frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\mu}}.$$

It must be noted that the motion of P is purely fictitious, and is simply a geometrical device for obtaining a motion of the point N, which will be Harmonic.

If the motion of N is to be timed from some other position than A such as N′, then, when $t = 0$, the corresponding position of P on the circle will be P′.

Let AOP′ $=\varepsilon$; then AOP $=\omega t+\varepsilon$ and $x=a\cos(\omega t+\varepsilon)$ or $x=a\cos(\sqrt{\mu}\,t+\varepsilon)$.

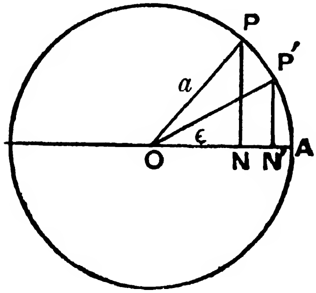

FIG. 149.

This angle ε is called the *epoch*, and the whole angle $\sqrt{\mu}\,t+\varepsilon$ is known as the *phase*. The terms are often used in Astronomy.

If we differentiate this value of x with respect to t, we get

$$\dot{x}=-a\sqrt{\mu}\sin(\sqrt{\mu}\,t+\varepsilon),$$

and $$\ddot{x}=-a\mu\cos(\sqrt{\mu}\,t+\varepsilon)=-\mu x,$$

i.e. $x=a\cos(\sqrt{\mu}\,t+\varepsilon)$ is also a solution of the equation $\ddot{x}+\mu x=0$.

NOTE. The direct solution of $\ddot{x}+\mu x=0$ is effected by multiplying throughout by $2\dot{x}$.

Then $$2\dot{x}\ddot{x}+2\mu x\dot{x}=0,$$

$$i.e.\quad \frac{d}{dt}(\dot{x}^2)+\mu\frac{d}{dt}(x^2)=0.$$

$$\therefore\ \dot{x}^2+\mu x^2=c.$$

If $\dot{x}=0$ when $x=a$, we have $0+\mu a^2=c$.

$$\therefore\ \dot{x}^2=\mu(a^2-x^2)\quad\text{or}\quad \dot{x}=\pm\sqrt{\mu}\sqrt{a^2-x^2}.$$

If $$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{a^2-x^2},$$

$$\therefore\ dt = -\frac{dx}{\sqrt{\mu}\,\sqrt{a^2 - x^2}}.$$

Put $x = a\cos\theta$. $\quad\therefore\ dx = -a\sin\theta\,d\theta.$

$$\therefore\ dt = +\frac{a\sin\theta\,d\theta}{\sqrt{\mu}\,a\sin\theta} = \frac{1}{\sqrt{\mu}}\,d\theta.$$

$$\therefore\ t = +\frac{1}{\sqrt{\mu}}\,\theta + c'.$$

If $x = a$ when $t = 0$, $\qquad c' = 0.$

$$\therefore\ \sqrt{\mu}\,t = \cos^{-1}\frac{x}{a}. \quad \therefore\ x = a\cos\sqrt{\mu}\,t.$$

EXAMPLE 1. A mass of 10 lbs. hangs at the end of a light spiral spring which extends 2″ for each pound it supports. Find the time of oscillation of the weight if it is pulled down and then released.

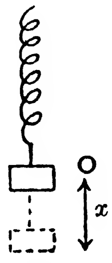

FIG. 150

Suppose the weight to be x'' from its equilibrium position, then the resultant force towards O will be kx. When $x = 2''$, this force is 1 lb.

$$\therefore\ 1 = k2. \qquad \therefore\ k = \tfrac{1}{2}.$$

The equation of motion is $\dfrac{\frac{1}{2}x}{10} = -\dfrac{\frac{\ddot{x}}{12}}{g}$

$$\therefore\ \ddot{x} + \frac{12g}{20}x = 0. \quad \therefore\ T = 2\pi\sqrt{\frac{5}{3g}}.$$

The forces acting may be clearer if the question is discussed at greater length.

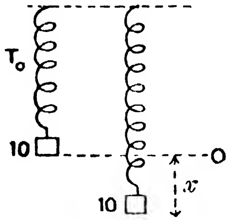

FIG. 151.

Let T_0 be the tension in the spring when in the equilibrium position; then $T_0 = 10$.

When the spring is extended a further distance x, let T be the tension; then $T - T_0 = kx$, but $T_0 = 10$. $\therefore T - 10 = kx$.

The equation of motion of 10 lbs. is

$$\frac{T-10}{10} = -\frac{\ddot{x}/12}{g},$$

$$i.e. \quad \frac{kx}{10} = -\frac{\ddot{x}/12}{g}, \text{ as before.}$$

EXAMPLES XXXV.

1. A weight hangs from a fixed point by a light spiral spring which it stretches from its natural length of 6″ to 8″ when the weight hangs at rest. The weight is pulled down until the spring is 9″ long and then let go. Assuming the tension of the spring is proportional to the excess of its length over 6″, find the greatest velocity of the weight and the number of complete oscillations per minute.

2. A particle performs 150 complete oscillations per minute and its greatest acceleration is 10 ft./sec².; find (1) its greatest velocity, (2) its mean velocity during the motion from one extreme position to the other.

3. A light helical spring extends 0·5″ when carrying a load of 5 lbs. How many vibrations per sec. will it make when carrying a load of 10 lbs. ?

4. A weight of 2 lbs. at the end of an elastic string passes from its highest position to its lowest, a distance of 6″, in $\frac{1}{4}$ sec. Find the tension of the string when the weight is at its lowest point.

5. A flat plate with bodies resting upon it begins to oscillate vertically through a distance of 4″. Determine within what limit the number of vibrations per min. must be kept if the bodies upon the plate are not to be thrown off by the vibration.

6. A shelf oscillates vertically with S.H.M. of period 0·5 sec. Show that if the amplitude is 2·5″ an object on the shelf will leave it when it is nearly at the highest point of its path.

7. A boy weighing 6 stone standing on a plank oscillates vertically in S.H.M of amplitude 6″ and period 1 sec. Find the greatest and least pressures he causes on the plank.

8. A mass of 1 lb. is hung to a light spiral spring and produces a static deflection of $1\frac{1}{2}$″. A mass of 1 lb. is suddenly added to the original mass. Find the maximum elongation produced. Find the time of oscillation of the whole mass.

9. A uniform heavy sphere weighing 32 lbs., radius 3″, is suspended by a wire from a fixed point, and the torsion couple of the wire is proportional to the angle through which the sphere is turned from the equilibrium position. If the period of oscillation is 2 secs., find the couple which will hold the sphere at rest when it is turned through 360° from its equilibrium position.

10. A mass of 5 lbs. is hung on to a light spring and is found to stretch it 4″ ; it is then pulled down a further 2″ and released. Find its K.E. when passing through the position of equilibrium (1) by finding the velocity, (2) by finding the work done by the spring.

11. A point is moving in a straight line with S.H.M. Its velocity has the values 3 ft./sec. and 2 ft./sec. when its distances

from the mean position are 1 ft. and 2 ft. respectively. Find the length of its path and the period of its motion. Find also what fraction of the period is occupied in passing between the specified points.

12. A crank OP 1 ft. long rotates round O and makes 120 revolutions per minute. It is driven by a piston and a connecting rod sufficiently long for the motion of the piston to and fro to be taken as Harmonic. When the crank makes 45° with the direction of the piston, find the force necessary to accelerate the reciprocating parts if their total mass is 200 lbs.

The Simple Pendulum.

When a small body, all of whose mass may be supposed collected at one point, swings at the end of a cord whose other end is fastened to a fixed point, it is called a Simple Pendulum.

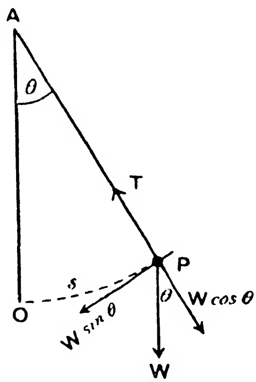

FIG. 152.

Suppose the string of length l to be deflected at an angle θ to the vertical.

The component of the weight at right angles to PA is W $\sin\theta$.

If s be the distance of P from O measured along the arc from O, the acceleration of P is $\frac{d^2s}{dt^2}$ or $\ddot{s}$, and is positive in the direction O to P.

The equation of motion will be

$$\frac{W \sin \theta}{W} = -\frac{\ddot{s}}{g}.$$

If θ is small, we may write θ for $\sin \theta$, and since $\theta = \frac{s}{l}$, the equation becomes

$$\frac{s}{l} = -\frac{\ddot{s}}{g} \quad \text{or} \quad \ddot{s} + \frac{g}{l} s = 0,$$

an equation of the same form as that on p. 266, where $\mu = \frac{g}{l}$. If θ is small the motion is therefore Harmonic, and

$$\text{the Periodic Time} = 2\pi\sqrt{\frac{l}{g}}.$$

It will be noted that this time is independent of θ, so that the oscillations are isochronous, *i.e.* they take equal times for all values of θ, provided θ is small.

The tension in the string (T) will be given by the equation

$$\frac{T - W \cos \theta}{W} = \frac{v^2}{lg},$$

since the acceleration of P towards O will be $\frac{v^2}{l}$, if v is the velocity of P along the tangent when OAP is θ.

Experiment. Set up a simple pendulum and find the time of an oscillation by timing 20 swings.

(i) Show that for different initial values of θ the average time of an oscillation is constant.

(ii) Show that if the length of the string is altered $T \propto \sqrt{l}$.

(iii) Calculate the value of g.

Compound Pendulum.

Any solid body suspended from a horizontal axis is found to perform oscillations similar to those described by a simple pendulum. Such a body is termed a Compound Pendulum.

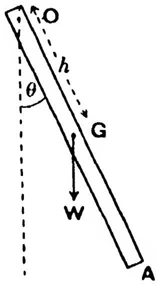

FIG. 153.

Suppose a rod of length $2a$ to swing in a vertical plane about a horizontal axis at one end.

When the rod makes an angle θ with the vertical the moment of the couple tending to bring it into the vertical position is $Wh \sin \theta$ if h is the distance of the C. of G. from O. (A vertical upward force equal to W acts on the rod at O.)

The equation of motion is

$$\frac{Wh \sin \theta}{Wk_1^2} = -\frac{\ddot{\theta}}{g},$$

where k_1 is the radius of gyration about O. (Note the negative sign since θ is diminishing.)

When θ is small this becomes

$$\frac{h\theta}{k_1^2} = -\frac{\ddot{\theta}}{g} \quad \text{or} \quad \ddot{\theta} + \frac{gh\theta}{k_1^2} = 0.$$

Hence the periodic time is

$$2\pi\sqrt{\frac{k_1^2}{gh}}.$$

If k is the radius of gyration about a parallel axis through G, we have by the parallel axis theorem $k_1^2 = k^2 + h^2$.

Simple Equivalent Pendulum.

The periodic time for a Simple Pendulum of length l is $2\pi\sqrt{\frac{l}{g}}$: therefore a simple pendulum will perform its oscillations in the same time as the Compound Pendulum, if

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{k^2+h^2}{gh}},$$

i.e. if $$lh = k^2 + h^2,$$

or $$h(l-h) = k^2, \textit{ i.e. } \mathrm{OG}\,.\,\mathrm{SG} = k^2.$$

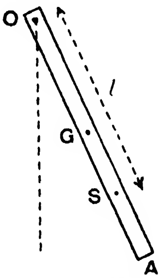

FIG 154.

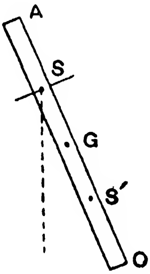

FIG. 155.

If we mark off from O a length OS $= l$, then GS $= l - h$.

If now we put GS $= h$, then OG $= l - h$, and we get

$$\mathrm{SG}\,.\,\mathrm{OG} = k^2,$$

i.e. S and O are interchangeable.

If then we put the axis at S, the pendulum would swing in the same time as it does when the axis is at O. If the pendulum is a uniform bar, it would obviously oscillate in the same time if the original axis were at A instead of at O.

At a distance l from A there will therefore be another

point S′ about which the time of oscillation is the same; there are therefore four positions for the axis about which the bar swings in equal times.

In order to calculate the value of g, Captain Cater used a compound pendulum consisting of a bar with two parallel knife edges, one of which was fixed near the end of the bar, while the other could be clamped at any point along the bar. The latter was adjusted until the time of swing about it equalled the time of swing about the fixed axis; the distance between the knife edges gave l, and from the formula $T = 2\pi\sqrt{\frac{l}{g}}$, g was calculated.

Experiment. Suspend a metal bar by a knitting needle passed successively through holes bored at intervals along the length of the bar.

Let the bar oscillate about the needle as an axis, and find the average time of a small oscillation. The following results were obtained:

Distance of the axis from one end O.	1·5	10	16	30	42	48	52	90	cms.
Time of oscillation.	2·02	1·94	1·91	1·88	1·92	1·96	2·0	2·01	secs.
Distance of the axis from one end O.	100	106	115	125	134	138	144	cms.	
Time of oscillation.	1·94	1·91	1·88	1·89	1·92	1·96	2·0	secs.	

Draw a graph of these results.

Find from them the length of the S.E.P.

Where approximately is the C. of G. of the rod?

Calculate the value of g.

Simple Equivalent Pendulum by the Energy Principle.

Let the Compound Pendulum start swinging from the position in which $\theta = \alpha$, and let its angular velocity be ω when the angle it makes with the vertical is θ. If k_1 is the radius of gyration about O we have by the Principle of Energy,

$$W' k_1^2 \frac{\omega^2}{2g} = W' h(\cos\theta - \cos\alpha),$$

since G has descended a vertical distance $h(\cos\theta - \cos\alpha)$.

$$\therefore\ k_1^2 \frac{\omega^2}{2g} = h(\cos\theta - \cos\alpha). \quad\text{.................(i)}$$

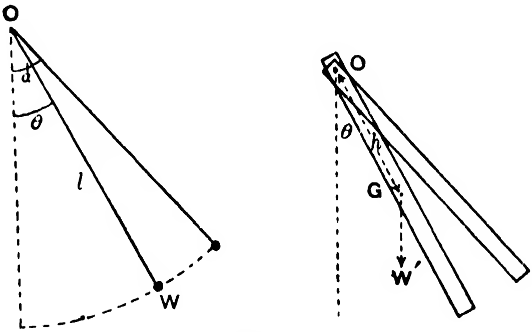

FIG. 156.

If the S.E.P. keeps pace exactly with the compound pendulum we have

$$W \frac{v^2}{2g} = W l(\cos\theta - \cos\alpha).$$

But if ω is the angular velocity of W about O, $\omega = \frac{v}{l}$.

$$\therefore\ \frac{l^2\omega^2}{2g} = l(\cos\theta - \cos\alpha). \quad\text{.................(ii)}$$

Comparing this equation with (i), we have

$$\frac{k_1^2}{h} = l, \quad\text{or}\quad k_1^2 = hl.$$

$\therefore\ k^2+h^2=hl$, if k is the radius of gyration about G. This is the same result as that obtained on p. 276.

Since $\dfrac{dl}{dh}=1-\dfrac{k^2}{h^2}$ it follows that if $h=k$. a small error in h will have little effect on the time of an oscillation, since the necessary change in l will be negligible.

EXAMPLES XXXVI.

1. Calculate in cms. the length of a simple pendulum which beats seconds, *i.e.* its periodic time $=2$ secs.

2. Find the period of vibration of a thin rod 100 cms. long suspended by one end and oscillating under the influence of gravity.

3. A thin rod OA, 2 ft. long, is suspended at O and is fixed at A to the rim of a circular disc of diameter 1 ft., so that OA produced passes through the centre. Find the time of a small oscillation in the plane of the disc, neglecting the mass of the rod.

4. A light flat spring is clamped horizontally at one end and a mass is attached to the other end; it is found that the mass deflects the spring a small distance d. Show that the time of oscillation about the position of equilibrium is the same as that of a pendulum of length d.

5. Find the times of a complete oscillation of a uniform circular disc suspended so as to swing about an axis, (1) through a point in the circumference perpendicular to its plane, (2) about a line through a point in the circumference tangential to the disc. The diameter of the disc is equal to the length of the seconds pendulum. (C.S.C.)

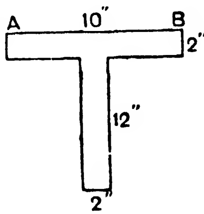

FIG. 157.

6. A piece of wood of uniform thickness is in the shape of a T. Find the time of oscillation in its own plane about the mid point of AB. If it is turned until AB is vertical and then released, find the angular velocity when it passes the equilibrium position.

7. A flywheel weighing 10,000 lbs. is suspended from a pair of centres entering conical holes in the rim, so that it can swing in a vertical plane. The line joining the centres is parallel to and distant 3 ft. from the axis of the wheel. The period of a complete swing is 2·5 secs. Find its radius of gyration about its axis and its K.E. when running at 250 revs. a min.

8. An irregular bar hangs with the points A and B in a vertical line and $AB = c$. When it swings about a knife edge at A the length of the S.E.P. $= c - a$; when it swings about B in an inverted position the length of the S.E.P. $= c - b$. Prove that $AG = \frac{bc}{a+b}$, and find the value of the radius of gyration of the bar.

9. When a wire is fixed at one end O, the torque required to twist it through one radian at unit distance from O is k, and to twist it through θ radians at a distance l from O is $\frac{k\theta}{l}$. If a rod suspended from the wire at its mid points is twisted and then let go, find the time of an oscillation, given that the M.I. of the rod about its mid point is **I**.

10. A particle is fastened to the mid point P of an elastic string fixed at its ends to two points A and B on a horizontal table. If P is pulled a distance x at right angles to AB, prove that AP now equals $l\left(1 + \frac{x^2}{2l^2}\right)$ nearly, where $AB = 2l$. Assuming $\frac{x}{l}$ is small, show that the equation of motion if the particle is released is approximately $\ddot{x} + \frac{2Tg}{Wl}x = 0$, where T is the initial tension in the string, and find the time of a small oscillation.

11. The ball of a simple pendulum weighs 50 grams and the length of the pendulum (measured to the centre of the ball) is 50 cms. If the ball is released when the cord is horizontal, find the tension in the string when it has swung through 45°, and find also the tangential acceleration of the ball at that instant.

12. A rod of length $2a$ is suspended by two vertical strings of length l at its ends, and is twisted through an angle θ about

its mid point G, which rises vertically. If ϕ is the angle each string now makes with the vertical, prove $\phi = \frac{a}{l}\theta$ if θ is small.

Hence find the time of an oscillation if the rod is released.

Compound Harmonic Motion.

Fasten a string ACB to two points A and B in a horizontal line, and tie another string, carrying a heavy weight D, to ACB at its mid point C. If D is drawn aside in the plane ACB, it will swing when released as a pendulum of length CD. If displaced in a plane perpendicular to ACB it will swing as a pendulum of length ED.

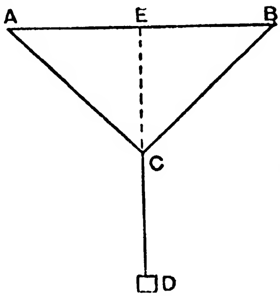

FIG. 158.

Let D swing in the plane ACB. and when it is passing the vertical position give it a blow at right angles to this plane. The resulting motion will be compounded of two Harmonic Motions at right angles to one another.

If D is a box filled with sand, having a small hole at the bottom, the curve traced out by D will be shown on a sheet of paper placed below it.

Another way of obtaining such a locus is to fasten a rod carrying a weight to the lower surface of a board which is supported so that it swings with the weight and moves harmonically.

A pen attached to another rod which also swings harmonically in a plane at right angles to the first pendulum is adjusted so as to trace a curve on a sheet of paper carried on the board. The curve described by the pen will be due to the composition of two Harmonic Motions.

Geometrical Construction.

Let the Harmonic Motion in the direction OX be represented by the motion of a point along AB; this will be given geometrically by the foot of the perpendicular on AB from a point whose successive positions round the circle are 1, 2, 3, etc. If the Harmonic Motion in the direction OY has the same amplitude and begins at A′ in the direction A′B′, the successive positions will be given by the perpendiculars from 1, 2, ... on A′B′.

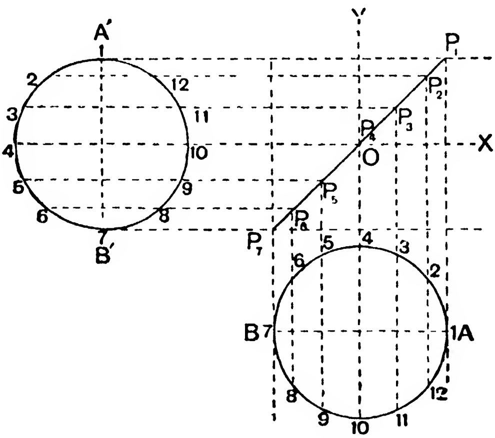

FIG. 159.

At the beginning of the motion the point which has both these motions will be at P_1, at intervals of $\frac{1}{12}$ of the periodic time it will be successively at P_2, P_3, etc., and the path of the point will be a straight line.

If the motion parallel to OY begins at 4 when the motion parallel to OX begins at 1, the curve is a circle. If the motion parallel to OY begins at 3 or at 5 when OX begins at 1, the curve is an ellipse.

Other curves will be produced if we make the periodic times of the two motions unequal. For instance, we can make the motion parallel to OY such that the relative speeds of the imaginary points travelling round the circles are in the ratio 2 : 3 (a fifth) or 2 : 1 (an octave).

Analytical Investigation.

Let the x coordinate of the point be given by $x=a\cos\sqrt{\mu}\,t$ and the y coordinate by $y=b\sin\sqrt{\mu}\,t$, the origin being the mean position. In this case the amplitudes of the component vibrations will be a and b; the angular velocity of the imaginary points describing the circles will be $\sqrt{\mu}$, and when $t=0$, $x=a$ and $y=0$.

Since $\frac{x}{a}=\cos\sqrt{\mu}\,t$ and $\frac{y}{b}=\sin\sqrt{\mu}\,t$, we have $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and the resulting curve will be an ellipse. If $a=b$ it will be a circle.

CHAPTER XVI.

VECTOR MOMENTUM. TURBINES. COEFFICIENT OF RESTITUTION.

Momentum a Vector.

The momentum of a body of weight W moving with velocity v has been defined as $\mathrm{W}\frac{v}{g}$. Since velocity involves direction as well as magnitude, $\mathrm{W}\frac{v}{g}$ must also involve direction as well as magnitude, and is therefore a Vector.

If, then, a body of weight W has communicated to it a velocity $\vec{u}$ in one direction and a velocity $\vec{v}$ in some other direction, its resultant Momentum will be the resultant of $\mathrm{W}\frac{u}{g}$ and $\mathrm{W}\frac{v}{g}$ found by vector addition.

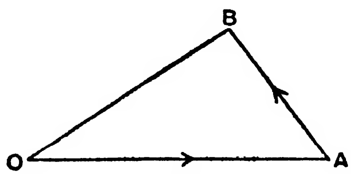

FIG. 160.

Let $\overrightarrow{\mathrm{OA}}$, $\overrightarrow{\mathrm{AB}}$ represent u and v in magnitude and direction respectively; then $\overrightarrow{\mathrm{OB}}$ represents the resultant velocity V.

On a different scale, if $\overrightarrow{OA}$ represents $W\frac{u}{g}$, then AB will represent $W\frac{v}{g}$ and $\overrightarrow{OB}$ will represent the resultant momentum $W\frac{V}{g}$.

If a body of weight W moving with velocity u receives an impulse Pt which would give the body, if at rest, a velocity v, then $Pt = W\frac{v}{g}$, so that AB represents this impulse and $\overrightarrow{OB}$ represents the resultant momentum of the body.

Experiment 1. Two steel balls, preferably with pointers attached to them, are suspended by long threads as in Expt., Pt. I. p. 118, so that they are just in contact, with the pointers close to a horizontal sheet of paper. A is drawn back to A_1 in a direction inclined to the line of centres, care being taken to move the thread in a vertical plane.

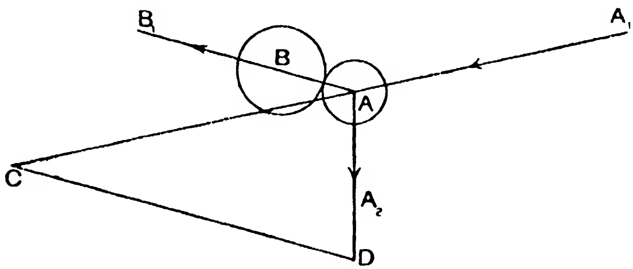

FIG. 161

When A is released it will strike B obliquely and will move to some position A_2, while B moves to B_1. The ball A weighed 35 grams and B 67 grams.

$A_1A = 16{\cdot}5$ cms., $AA_2 = 4{\cdot}5$ cms., $BB_1 = 7$ cms.

The velocity of A on impact is proportional to A_1A (see Pt. I. p. 110).

∴ the momentum of A is then proportional to

$$35 \times 16{\cdot}5 = 577.$$

After impact its momentum is proportional to $35 \times 4{\cdot}5 = 157$. The momentum given to B is proportional to $67 \times 7 = 469$.

Draw AC to represent 577, and draw AD to represent 157. Join DC; it will be approximately parallel to BB_1.

Since a momentum $\overrightarrow{AD}$ combined with a momentum $\overrightarrow{DC}$ equals a momentum $\overrightarrow{AC}$, it follows that the loss in A's momentum caused by the blow is $\overrightarrow{DC}$, and it will be found that $\overrightarrow{DC}$ represents 469 approximately. Hence the loss of momentum by A equals the gain in B's momentum.

Experiment 2. A glass tube bent at right angles at C passes through a cork at A which is fitted with a stout wire whose ends rest in fixed glass tubes so that the tube can oscillate in a vertical plane. A thread fastened to the tube at B passes over a pulley and supports a scale pan which is loaded until BC is vertical.

When water is supplied to the tube through a rubber tube at G, the momentum of the water in the direction BC will be destroyed at the bend and a horizontal momentum along CD will be created. The vertical impulse is taken by the pivot at A, and the horizontal reaction at C will swing the tube out of the vertical. Weights are added to the scale pan until BC is again vertical.

The following results were obtained by experiment :

$$BG = 6{\cdot}2 \text{ cms.}, \quad GC = 41{\cdot}7 \text{ cms.},$$
$$T = \text{tension in } BH = 98 \text{ grams wt.}$$

Taking moments about A we have, if P is the resultant horizontal reaction at the bend,

$$P \times 41{\cdot}7 = 98 \times 6{\cdot}2. \quad \therefore \; P = 14{\cdot}6 \text{ grams wt.} \quad \ldots\ldots\text{(i)}$$

The weight of water in CD would make the tube swing to the right, but if CD is small the effect is negligible.

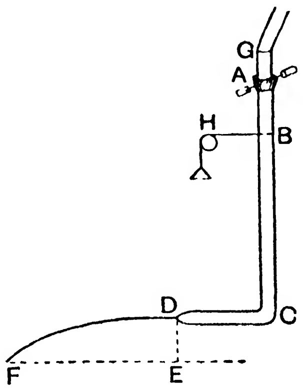

FIG. 162.

To find the change in momentum:

The jet of water fell 18·8 cms. (DE) in a horizontal distance of 56 cms. (EF). ∴ if v is the velocity of the water at D,

$$56 = vt \quad \text{and} \quad 18{\cdot}8 = \tfrac{1}{2} \cdot gt^2.$$

$$\therefore \; 18{\cdot}8 = \frac{981}{2} \cdot \left(\frac{56}{v}\right)^2. \quad \therefore \; v = 286 \text{ cms./sec.}$$

In 15 secs., 730 grams of water were collected.

∴ the horizontal thrust P of the tube at C produced a

$$\text{horizontal momentum} = W\frac{v}{g} = 730 \times \tfrac{286}{981}.$$

$$\therefore \; P \times 15 = 730 \times \tfrac{286}{981}.$$

∴ P = 14·3 grams wt., which agrees approximately with the value found in (i).

EXAMPLES XXXVII.

1. Water is flowing round a right-angled bend in a pipe. If 1 cwt. of water passes the bend each second with a velocity of 5 ft./sec., find the thrust produced on the pipe at the bend.

2. A body weighing 20 lbs. is moving horizontally at 10 ft./sec. A force constant in magnitude and direction acts upon it for 2 secs., after which the body is moving at the same speed in a direction inclined at 45° to its original direction. Find the magnitude of the force.

3. Sand is poured on to a smooth fixed plane surface inclined at 60° to the horizontal. The sand falls from rest through a distance of 10 ft. before striking the plane, and the grains do not rebound but run straight down the plane. If 1 lb. of sand be discharged per minute, show that the total pressure normal to the plane due to the falling sand is about $\frac{1}{10}$ oz. weight.

4. A straight rod carries two particles each of mass 40 grammes at its extremities A and B. The rod turns about a point C, where CA = CB = AB, at the rate of 2 revolutions per sec. If AB = 20 cms., find the magnitude and direction of the resultant of the momenta of A and B.

5. A cricket ball weighing $4\frac{1}{2}$ oz. reaches the batsman when travelling horizontally with a speed of 80 ft./sec. After the batsman has struck it the ball is travelling again horizontally with a velocity 90 ft./sec., and on the leg side in a direction making 45° with the direction of the bowler. If the bat is in contact with the ball for $\frac{1}{40}$ sec., find the average magnitude and the direction of the batsman's stroke.

6. A rifle is fixed to a heavy block which can swing about a fixed horizontal axis, the line of the barrel being at right angles to the axis. The discharge of the rifle produces such recoil that the block swings through an angle θ. If in a series of experiments the same bullet is used but with different charges of powder, prove that the muzzle velocity of the bullet is proportional to $\sin\frac{\theta}{2}$.

7. A block of wood weighing 10 lbs. rests on a horizontal table on which it can move freely in any direction. A bullet weighing $\frac{1}{30}$ lb. is fired horizontally into the block with a velocity of 1500 ft./sec. The bullet passes through the block, but is deflected from its course, its direction of motion after emerging being horizontal but inclined at 30° to the line of firing. The velocity of the bullet after emerging is 1200 ft./sec. Find the magnitude and direction of the velocity of the block just after the bullet has emerged. (C.U.)

8. A jet of water issues vertically at 30 ft./sec. from a nozzle 0·1 sq. in. in section. A ball weighing 1 lb. is balanced in the air by the impact of the water on its under side. Show that the height of the ball above the level of the jet is 4·6 ft. approx.

9. If an impulse $\mathbf{I}$ changes the velocity of a body from u to v and E is the change in K.E., prove that $\mathsf{E} = \mathbf{I}\left(\frac{u+v}{2}\right)$.

10. Two railway trucks each of mass 5 tons are fitted with inelastic buffers and are coupled by a chain allowing 6″ of slack. They are running at 8 ft./sec. with the coupling taut when the first truck reaches a pair of spring buffers which would stop either truck when running at that speed in 6″. Find the speed at which the trucks leave the stop on the rebound, assuming the pressure exerted by the spring buffers is constant. (C.U.)

11. A gun weighing W lbs. fires a shot of w lbs., and is free to recoil on a horizontal plane. If the gun is elevated at an angle α and recoils with velocity u, prove that the velocity of the shot relative to the barrel is $\frac{(\mathsf{W}+w)u}{w\cos\alpha}$.

12. A gun free to recoil horizontally is fired at an angle of elevation α. The shot leaves the gun at an elevation β. If the weight of the gun is W and of the shot w, prove that

$$\tan\alpha = \frac{\mathsf{W}\tan\beta}{\mathsf{W}+w}.$$

13. Prove that a gun of weight W suspended at an elevation α by equal parallel ropes will send a shot of weight w to a range

of $4\frac{W}{w}\left(1+\frac{W}{w}\right)h\tan\alpha$ if h is the vertical height ascended in the recoil. Find the change in the tension of a rope.

14. A bullet of mass m is fired horizontally into a block of wood of mass M which is free to move on a smooth horizontal table, and penetrates it to a distance a. Show that at the instant when the bullet comes to rest relatively to the block, the block has moved a distance $\frac{ma}{M+m}$; the stress between the bullet and block being assumed constant so long as there is any relative motion.

15. A train consists of an engine of mass M tons and 2 wagons of mass m tons each. At the start the buffers are in contact, and when the coupling chains are tight the buffers are a feet apart. The train starts with the engine exerting a constant tractive force of F tons wt. Neglecting friction, prove that the second wagon starts with velocity v ft./sec., where

$$v^2=\frac{2Fga(2M+m)}{(M+2m)^2}. \qquad \text{(C.U.)}$$

16. A shell is projected into the air in any direction; at a point in its path it breaks up into two equal parts, one of which flies vertically upwards; prove that the other part will proceed to describe a parabola whose S.L.R. is four times as large as that of the first parabola.

The Principles of Momentum applied to Fluid Pressure. Efficiency of Poncelet's Water-wheel.

The wheel is rotated by the pressure of the water against the blades. The velocity of the water is reduced, and its depth increased after passing the wheel. The change per sec. in the momentum of the water measures the force on the wheel (see Pt. I. p. 128). Let V ft./sec. be the velocity of the water on reaching the wheel, and v the velocity on leaving it. If the blades fit the gully, the velocity

of the blades will be practically equal to v. Since the blades are small, the velocity of the whole blade may be taken as equal to v, the velocity of the extremities.

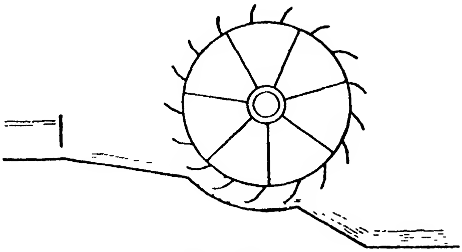

FIG. 163.

If A is the area of the cross-section of the gully in sq. ft., in each second AV cu. ft. of water weighing AV (62·3) lbs. reach the wheel with velocity V, and the same amount leaves with velocity v.

$$\therefore \text{ change of momentum per sec.} = \mathrm{AV}(62\cdot 3)\left(\frac{\mathrm{V}-v}{g}\right)$$

$$= \text{pressure on the blades (in lbs. wt.).}$$

Since the blades are moving with velocity v,

$$\text{the work done per sec.} = 62\cdot 3\,\mathrm{AV}\,\frac{(\mathrm{V}-v)}{g}\,.\,v \text{ ft.-lbs.}$$

Now the K.E. of the water which reaches the wheel in each second is

$$\mathrm{AV}(62\cdot 3)\,.\,\frac{\mathrm{V}^2}{2g} \text{ ft.-lbs.}$$

$$\therefore \text{ efficiency of the wheel} = \frac{62\cdot 3\,\mathrm{AV}\,\dfrac{(\mathrm{V}-v)v}{g}}{62\cdot 3\,\mathrm{AV}\,\dfrac{\mathrm{V}^2}{2g}} = \frac{2(\mathrm{V}-v)v}{\mathrm{V}^2}.$$

If V is constant, this expression is a maximum when $\mathrm{V}v - v^2$ is a maximum.

This may be written

$$\frac{V^2}{4} - \left(\frac{V}{2} - v\right)^2,$$

and is therefore a maximum when $\frac{V}{2} - v = 0$, *i.e.* when $v = \frac{V}{2}$ $\left[\text{or, by differentiating, when } V - 2v = 0, \textit{ i.e. } v = \frac{V}{2}\right]$.

In this case efficiency $= \dfrac{2\left(\frac{V}{2}\right)\frac{V}{2}}{V^2} = \frac{1}{2}$.

Turbines.

Suppose a jet of water to be directed along a surface AB so that it enters tangentially to the surface at A with velocity v, and leaves horizontally at B. There will be no normal impulse on the water entering at A, and therefore no sudden loss of energy. The direction of the flow will be gradually changed, and if we ignore frictional losses it will leave at B with velocity v.

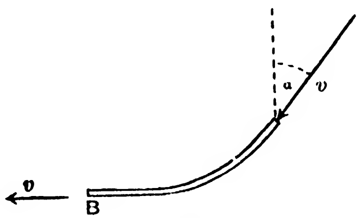

FIG. 164

Let W lbs. per sec. of water enter at A at an angle a to the vertical. Since the water leaves B with no vertical velocity the vertical momentum of the water is destroyed between A and B, *i.e.* $W\frac{v\cos a}{g}$ units of momentum will

be destroyed per sec. If P be the vertical pressure of the surface; $P \times 1 = W\frac{v \cos a}{g}$. $\therefore$ the vertical pressure of the water on the surface is $W\frac{v \cos a}{g}$.

The horizontal pressure will be measured by the change in horizontal momentum per sec., *i.e.* $W\frac{(v - v \sin a)}{g}$.

Suppose now the surface to be moving vertically downwards with velocity u. If the jet were kept at the angle a the water would no longer arrive tangentially to the moving surface. It must be directed so that its velocity *relative* to A is in the direction of the tangent at A, *i.e.* we must split v up into two components, one equal to u the velocity of A, the other component must be in the direction of the tangent at A.

Let $AC = u$; draw CD parallel to AT the tangent at A, and mark off $AD = v$; then AD, making an angle β with the vertical, will be the required direction for the jet.

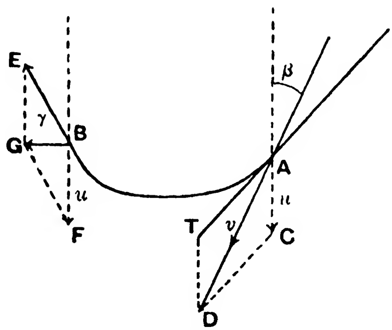

FIG. 165.

If the direction of the surface at B were horizontal, the water would no longer leave the surface horizontally, as

it would have the vertical velocity u of the surface. In order to make it leave horizontally the surface must be curved at B. Let V be the velocity of the water at exit, then at B it will have a velocity V in the direction BG, and u is its velocity vertically. The angle γ which BE makes with the horizontal is therefore given by $\tan\gamma = \frac{u}{V}$.

Now if W lbs. of water impinge on A in each second the water arriving at A has a momentum measured vertically of $W\frac{v\cos\beta}{g}$. When it leaves at B it has no vertical momentum. $\therefore$ the loss of momentum per second is $W\frac{v\cos\beta}{g}$, and this measures the vertical pressure of the water on the surface.

When A moves, the water delivered by a fixed jet will not strike the surface AB at A, but in practice AB is one of a series of blades close together, so that when one blade has moved a short distance, another receives the water from the jet.

In a turbine the shaft is made to rotate by the pressure of a jet of water or steam directed by guides against blades which are fixed to a wheel.

Let BA represent one of a series of curved blades fixed radially to a wheel, and suppose a jet of water or steam under pressure to be directed with velocity V in a direction AD so as to strike the blade when its extremity A is moving round O with velocity u.

If the direction AD is such that the jet has no relative velocity normal to the blade at A, there will be no impulse on the water from the blade on impact which would cause a sudden loss of K.E. to the jet. The velocity of the jet

relative to the blade on reaching it will then be entirely tangential to its surface.

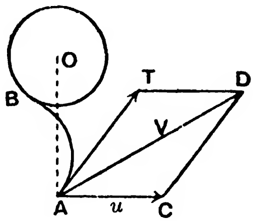

FIG. 166.

If AT is the direction of the tangent to the blade at A, we split V up into two components, one along AT and the other in the same direction as u and equal to u.

Draw AC $= u$ and CD parallel to AT. Cut off AD $=$ V, then AD is the required direction of the guide.

The velocity V may be resolved into two components at right angles to one another, V_t along AC and V_r along AO. (Note that in general, V_t is not equal to u.) Let the water reach and leave the blade at the rate of W lbs. per second. During its passage along the blade its direction will be gradually changed, and the blade is so shaped that the water is discharged at B into the hollow axis of the wheel with an absolute velocity directed towards O. The momentum of the water (measured in the direction AC), which arrives at the blade in a small interval of time δt is therefore $W\frac{V_t\,\delta t}{g}$. During the same interval of time the same amount of water is being discharged towards O in the direction AO, and this water has no momentum in the direction AC. A momentum of $W\frac{V_t\,\delta t}{g}$ in the direction AC is therefore being destroyed in time δt, which is at the rate

of $W\frac{V_t}{g}$ per second. This loss of momentum measures the force causing the rotation of the blade and $W\frac{V_t}{g}.AO$ measures the torque acting on the wheel.

The Thomson Turbine, which is one of inward radial flow, consists of an outer casing surrounding the guide chamber containing four guide blades (G) which direct

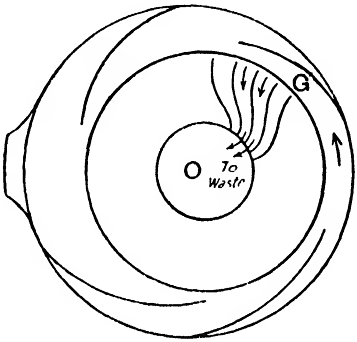

FIG. 167.

the water to the blades. The water from the chamber enters the apertures between the blades at openings all over the rim, but the angle at which it strikes the blades is practically the same as at the end of the guide blade.

The set of these guides depends upon the speed of the turbine and the water, and the guides are all pivoted so that they can be rotated from the outside.

In the de Laval turbine the blades are fixed on the rim of a wheel and project radially from it. Steam is directed on to the blades at one side of the wheel and leaves them on the other side.

The steam jet is directed so that it enters the side of the blades in a direction tangential to the surface of the moving

blades, and the blades are shaped so that the direction of the escaping steam is parallel to the axis of the wheel.

When water or steam enters the wheel at its outer rim and is ejected through the hollow axis of the wheel, the

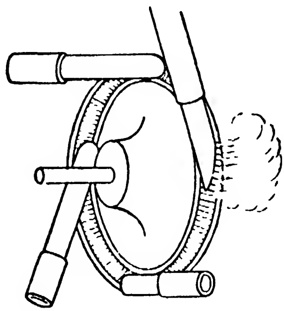

FIG. 168.

turbine is called an inward flow turbine. In the outward flow turbine the water enters at the axle and is discharged at the rim.

When water enters a wheel without pressure, as in the water wheel, it is called an Impulse Turbine. When it enters under pressure it is called a Reaction Turbine.

Moment of Momentum.

We have seen that if a body of weight W moving with velocity u in the direction AB is acted upon by an impulse Pt which causes the body to move in the direction AC with velocity v, the change in momentum is represented by BC and that BC represents Pt the impulse.

The three vectors, $W\frac{u}{g}$, $W\frac{v}{g}$, Pt being represented in magnitude and direction by the three sides of the triangle ABC, it follows, since AC is the resultant of the vectors

AB, BC, that the moment of $\overrightarrow{AC}$ about any point equals the algebraic sum of the moments of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ about

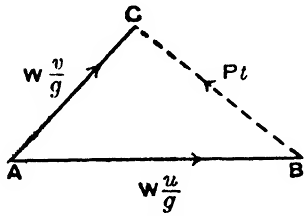

FIG. 169.

that point, the proof being identical with the corresponding statical theorem relating to Forces.

We have then the theorem that when an impulse changes the momentum of a body, the moment of the impulse about any point equals the difference between the moments of the final and initial momentum of the body about that point.

The moment of momentum of a body about any point is also known as the Angular Momentum of the body about that point.

Efficiency of a Turbine.

If we suppose the blade OBA to be small and consider that it is all moving with a velocity u, a simplified investigation to find the efficiency of a turbine may be made as follows :

Suppose that in each second a weight W lbs. of water reaches and leaves the blade, and that it arrives with a velocity whose component at right angles to AC is V_c and leaves in the direction OC.

The change in the momentum of the water at right angles to AC in 1 sec. is therefore $W\dfrac{V_t}{g}$, and this therefore measures the force on the blade.

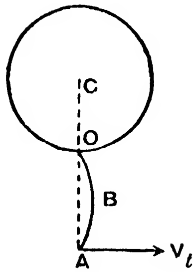

FIG. 170

This force acts through u feet in each second.

$$\therefore \text{ work done per sec.} = W\frac{V_t}{g}\,.\,u.$$

If the blade is shaped so that the tangent at A is radial, then $V_t = u$ if the water is directed so as to reach the blade without shock, and the work done per sec. is $W\dfrac{u^2}{g}$. The energy of the water arriving per sec. is $W\dfrac{V^2}{2g}$ if V is the actual velocity of the jet.

$$\therefore \text{ efficiency} = \frac{2u^2}{V^2}.$$

The following is a more detailed proof for the work done on the blade.

Suppose PR, P′R′ to be two positions of the blade rotating about O with angular velocity ω, and let a particle of water at P travelling along the moving blade reach Q in time δt. Its motion can be represented by supposing it to move from P to P′, where OP = OP′, with the blade, and then to move from P′ to Q along the surface of the blade. Let N be the pressure

along the normal CP per unit length of blade at P. Resolve N into $N \cos\phi$ along OP and $N \sin\phi$ perpendicular to OP

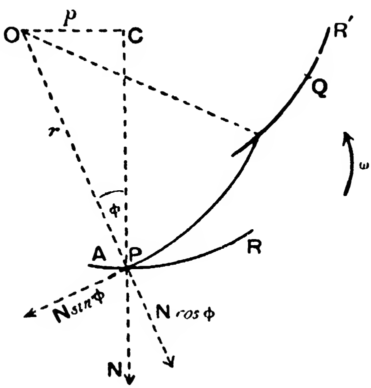

FIG. 171.

Since P moves at right angles to OP, $N\cos\phi$ does no work, while the work done by $N \sin\phi$ is $N \sin\phi \; \widehat{PP'} = N\sin\phi \; \omega r\,\delta t$ for

$P\hat{O}P = \omega\,\delta t$.

Let p be the perpendicular from O on the normal CP.

Work done by the water on length δs of the blade

$$= N\omega r \sin\phi \;\delta t\,\delta s = N\omega p\;\delta t\,\delta s.$$

$$\therefore \text{ total work done on the blade} = \Sigma N\omega p\,\delta t\,\delta s$$
$$= \omega[\delta t\,\Sigma N p\,\delta s]$$

$= \omega \times$ moment about O of the impulse of the water on the whole blade. ....(i)

If W is the weight of water entering the blade with velocity V_t measured at right angles to OA and leaving radially, the change of momentum in time δt is $W\dfrac{V_t\delta t}{g}$, and the change in the moment of momentum about O is $W\dfrac{V_t\,\delta t\,a}{g}$, where $OA = a$.

But the change in the moment of momentum about O = moment of the impulse about O.

$$\therefore\; W\frac{V_t\,\delta t\,a}{g} = \frac{\text{Total work done on the blade}}{\omega} \text{ (from (i)).}$$

But $a\omega = u$, the velocity of the end A of the blade.

$$\therefore \text{ work done on the blade in time } \delta t = \mathrm{W}\,\frac{\mathrm{V}_t u \,.\, \delta t}{g}.$$

$$\therefore \text{ work done per sec.} = \mathrm{W}\,\frac{\mathrm{V}_t u}{g}, \text{ as on p. 299.}$$

EXAMPLES XXXVIII.

1. The square sail of a ship has an area of 150 sq. ft. and the wind is blowing at right angles to the sail at 20 mls./hr. If the ship is moving at 6 mls./hr., find the pressure on the sail, given 1 cu. ft. of air weighs 0·081 lb., and assuming that after striking the sail the wind slides without friction along its surface.

2. In an inward flow turbine making 10 revolutions per sec., the internal diameter is 9″. Find the angle of the vanes at exit if the water leaves the wheel radially with a speed of 8 ft./sec.

3. The rim velocity of a turbine wheel is 50 ft./sec. and the blades are radial. If the jet is directed to reach the rim of the wheel at 70° to a radius, find the proper velocity of the jet and its initial speed along the radius.

4. A jet of steam whose velocity is 2000 ft./sec. is directed against the moving blades of a turbine at an angle of 15° to the direction of motion of the blades, which are shaped so that the steam leaves them radially. Find the force rotating the blades if 1 lb. of steam emerges from the jet in 2 minutes.

5. If the blades of the turbine in Qu. 4 are supposed to move with a speed of 1000 ft./sec. along the circumference of a circle, find the H.P. of the engine. Find also the efficiency of the turbine.

6. A turbine whose blades are radial at the outer rim receives 40,000 lbs. of water per min.; the outer radius of the wheel is 1 ft. and the H.P. is to be 50. How many revolutions per min. must the wheel make?

7. A jet of water of cross-section 1 sq. inch, moving horizontally at 30 ft./sec., strikes a smooth curved surface along a tan-

gent at a point A. The surface is concave towards the jet and is curved so that the tangent at B when the water leaves the surface is inclined at 45° to the tangent at A. If the surface is moving horizontally in the direction of the jet at 10 ft./sec., find the force on the surface in the direction of motion and also at right angles to that direction. Calculate the work done per sec. on the surface.

8. The blades of an inward flow turbine are radial and the rim moves at 30 ft./sec. Water enters the rim without shock where the radius is 2 ft. with a radial speed of 5 ft./sec. and leaves where the radius is 6″. The total area of the openings by which the water arrives is 2 sq. ft. Find the weight of water entering the turbine per sec. and the turning moment on the wheel.

9. A jet of water moving with a velocity of 24 ft./sec. along AB strikes a vane which is moving in the direction AC with a velocity of 12 ft./sec. If $B\hat{A}C = 30°$, find the angle between AC and the tangent to the vane at A if the water is to arrive without shock. Find also the angle between the tangent to the vane at its other extremity and the direction of its motion (parallel to AC) if the water is to leave it at right angles to AC. What is the pressure on the vane in the direction AC caused by each pound of water.

10. The moment of momenta of a particle of mass m about two points A and B in the plane of motion are h_1 and h_2. Show that the velocity of the particle perpendicular to AB is

$$\frac{(h_1 - h_2)}{m\mathrm{AB}} g. \qquad \text{(C.U.)}$$

Coefficient of Restitution.

When bodies made of different materials are dropped on to a horizontal surface from a given height, the height to which they will rebound varies with the material of which the bodies are made. Since they all reach the surface with the same velocity, it is clear that they leave it with different velocities.

Experiment. A steel ball is let fall from A on to a plate of steel B, and the height BC to which it rebounds is noted. The following results were obtained :

$$AB = h_1 \text{ cms.} = 64 \text{ cms.}$$
$$BC = h_2 \text{ cms.} = 41 \text{ cms.}$$

Now the velocity on reaching B is given by $v_1^2 = 2gh_1$, and the velocity on leaving is given by $v_2^2 = 2gh_2$.

$$\therefore \frac{v_2}{v_1} = \sqrt{\frac{h_2}{h_1}}.$$

Calculating these ratios, we have

$$\frac{v_2}{v_1} = \sqrt{\frac{41}{64}} = 0{\cdot}8.$$

For different values of v_1 and v_2, we find that $\frac{v_2}{v_1}$ always equals 0·8. Hence we see that for the materials steel and steel the velocity after impact bears a constant ratio to the velocity before impact and is in the opposite direction.

If the surface of the steel plate is chalked or greased, the ball will leave a circular mark showing the area which has been in contact with the plate, so that the ball must have been compressed and has regained its shape. The property which bodies have of regaining their shape is called elasticity.

The more general case of impact between two bodies was investigated by Newton, whose experiments were conducted in a way similar to that given in Pt. I. p. 119.

Taking the results obtained in that experiment, we have :

Velocity of A relative to B before impact is proportional to $+18{\cdot}9 = k(18{\cdot}9)$, taking measurements to the right as positive.

Velocity of A relative to B after impact

$$=(\text{A's velocity} - \text{B's velocity}) = k(8{\cdot}6 - 24)$$
$$= -k15{\cdot}4.$$

$$\text{The ratio } \frac{\text{Relative velocity after impact}}{\text{Relative velocity before impact}} = -\frac{15{\cdot}4}{18{\cdot}9} = -0{\cdot}8.$$

If A is drawn back different distances so as to vary its velocity on impact, we shall find that for two balls of steel the ratio of the relative velocities before and after impact $= -0{\cdot}8$.

This ratio is called the coefficient of restitution, and is found to be constant for bodies made of any two given substances. It is generally represented by the letter e.

Newton thus stated his law of impact :

The relative velocity after impact bears a ratio to the relative velocity before impact which is independent of the masses and the velocities before impact, and depends only on the nature of the balls. It is generally applied in the form :—

Relative velocity after impact $= -e$ (Relative velocity before impact).

The negative sign shows that the relative velocity after impact is in the opposite direction to that before impact.

A substance for which the coefficient of restitution is unity is said to be perfectly elastic ; no such substance exists although some approach it very nearly, the value of e for two balls made of glass being 0·94. Other values for e obtained by using both balls of the same material are ivory 0·8, cast iron 0·66, cork 0·65, lead 0·2. e, of course, depends upon the nature of each body ; it would be different for different specimens of the same material and different for steel in contact with glass and steel in contact with steel.

For glass and clay e would be practically zero. When e is approximately zero for two bodies of the same material they are said to be inelastic.

EXAMPLE. A body A weighing 4 lbs., moving at 8 ft./sec., collides with another B, weighing 6 lbs., moving in the same straight line at 6 ft./sec. If $e=0{\cdot}5$, find the velocities after impact if they were originally moving (1) in the same direction, (2) in opposite directions.

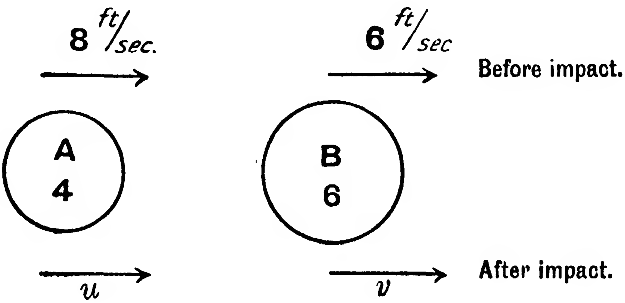

FIG. 172.

(i) Let u and v be the velocities of A and B respectively after impact measured positive to the right.

By conservation of momentum,

$$4\frac{8}{g}+6\frac{6}{g}=4\frac{u}{g}+6\frac{v}{g}.$$

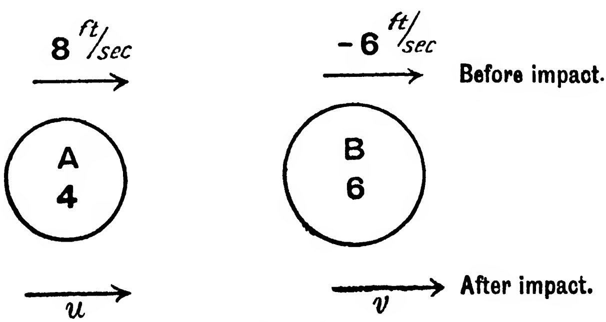

FIG. 173.

By Newton's Law,

$$v - u = -\tfrac{1}{2}(6 - 8).$$

From which $v = 7{\cdot}2$ ft./sec. $u = 6{\cdot}2$ ft./sec.

(ii) By conservation of momentum,

$$4 \cdot \frac{8}{g} - 6 \cdot \frac{6}{g} = 4\frac{u}{g} + 6\frac{v}{g}.$$

By Newton's Law,

$$v - u = -\tfrac{1}{2}(-6 - 8).$$

From which $u = -4{\cdot}6$ ft./sec. $v = 2{\cdot}4$ ft./sec.

Hence after impact B moves to the right and A moves to the left.

Neglect of Finite Forces during Impact.

The forces brought into play during impact are so large and the time is so short that the action of finite forces, such as gravity or frictional resistance, is usually ignored, since they can have no appreciable effect on the change of momentum. If, for instance, in Ex. 5, p. 124, the time t be 0·01 sec. the complete equations will be

$$(\mathsf{P} - 1)\frac{1}{100}\, 1 \cdot \frac{v}{g} \quad \text{and} \quad (\mathsf{P} - 2)\frac{1}{100} = 2\left(\frac{96}{g}\right) - 2\frac{v}{g}.$$

Hence $\mathsf{P} = 201\tfrac{1}{3}$ lbs. and $v = 64{\cdot}1$ ft./sec., instead of $\mathsf{P} = 200$ lbs. wt. and $v = 64$ ft./sec. If t is 0·001 secs., $v = 64{\cdot}01$ ft./sec.

Oblique Impact on a Fixed Surface.

If a body A strikes a surface B with velocity u at an angle α to the normal BN, and rebounds with velocity v at an angle β to the normal, we have the following equations, assuming the bodies are smooth :

(i) $u \sin \alpha = v \sin \beta$, since there is no tangential force to affect the velocity at right angles to the normal.

By Newton's Law,

Velocity of A − vel. of B = − e (vel. of A − vel. of B),
after impact before impact

$$i.e.\ v\cos\beta - 0 = -e(-u\cos\alpha - 0),$$

the direction BN being considered positive.

(ii) $$\therefore\ v\cos\beta = eu\cos\alpha.$$

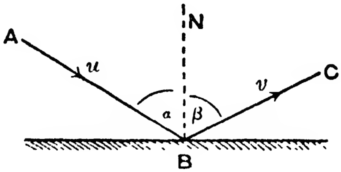

FIG. 171.

Dividing (i) and (ii), we have $\tan\alpha = e\tan\beta$.

If $e = 1$, α will equal β.

The impulse of the blow

$$= W\frac{[v\cos\beta - (-u\cos\alpha)]}{g} = W\frac{u\cos\alpha(1+e)}{g}.$$

Oblique Impact of two Smooth Spheres.

Let A and B be the centres of the spheres when impact occurs, and let A before impact travel along A_1A with velocity u_1, making an angle α with AB. Suppose that after impact A travels along AA_2 with velocity v_1, where $A_2AX = \beta$ and that B travels along BB_1 with velocity v_2.

Since there has been no impulse at right angles to AB, the velocity of A in this direction must be unaltered.

$$\therefore\ v_1\sin(\alpha+\beta) = u_1\sin\alpha.$$

In the direction AB the relative velocity of B to A after impact $= v_2 - v_1\cos(\alpha+\beta)$, and before impact $= 0 - u_1\cos\alpha$.

$\therefore$ by Newton's Law $v_2 - v_1\cos(\alpha+\beta) = -e(0 - u_1\cos\alpha)$.

By the conservation of momentum, the total momentum in the direction AB is unaltered, so that if W_1 be the weight of A and W_2 the weight of B,

$$W_2 \frac{v_2}{g} + W_1 \frac{v_1 \cos(a+\beta)}{g} = W_1 \frac{u_1 \cos a}{g}.$$

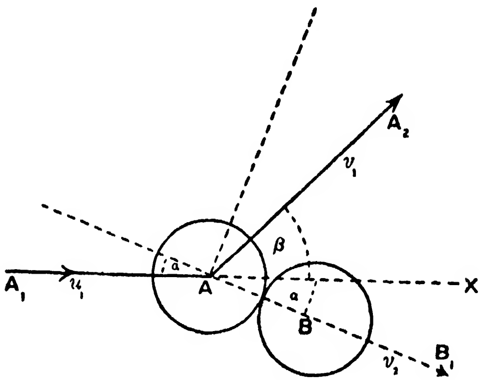

FIG. 175.

In practice the roughness of the surfaces sets up rotation when the impact is oblique and the examples are merely useful as theoretical exercises.

EXAMPLES XXXIX.

1. A ball weighing 5 lbs., moving at 8 ft./sec., catches up another ball weighing 4 lbs., moving in the same direction at 4 ft./sec. If the velocity of the latter after impact is 6·5 ft./sec., find the velocity of the other ball, the coefficient of restitution and the loss of K.E.

2. A sphere weighing 4 lbs., moving at 8 ft./sec., collides with another sphere weighing 1 lb., moving in the opposite direction at 4 ft./sec. If the latter recoils at 8 ft./sec., find e and the velocity of the larger sphere after impact.

3 A ball strikes a smooth plane, and its direction makes an angle of 40° with the normal before impact and 50° after impact. Find the value of e.

4. A smooth sphere weighing 10 lbs. is at rest when it is struck by another one weighing 5 lbs., moving at 20 ft./sec. in a direction making 45° with the line of centres at the moment of impact. If $e=0\cdot3$, find the subsequent motion of the spheres.

5. A body of weight W_1, moving with velocity u_1, catches up another body of weight W_2, moving in the same direction with velocity u_2. If the common velocity at the instant of greatest compression is u, find $\mathbf{I}_1$, the impulse up to that time, and if $\mathbf{I}_2$ is the impulse from that time until contact ceases, prove $\mathbf{I}_2=e\mathbf{I}_1$. If E_1 is the loss of K.E. during the first part and E_2 the gain of K.E. during the second part, prove that $E_2=e^2 E_1$.

6. Three equal smooth balls lie on a smooth table. A and B are in contact, and C is projected in a direction very nearly along the common tangent at the point of contact. It strikes A and immediately afterwards B. Prove that the ratio of the velocities of A and B after contact is $\frac{4}{3-e}$, where e=coefficient of restitution.

7. A particle is projected up an inclined plane of coefficient of elasticity e. If it rebounds perpendicularly to the plane after the second impact, show that $\cot\alpha=2(1+e)\tan\beta$; α=angle of plane and β the angle the direction of projection makes with the plane.

8. A ball A is projected directly towards another ball B with velocity v at elevation α, whilst at the same instant B is let fall. Prove that the balls will impinge, and that after impact, if $e=1$, A will fall vertically and B will describe a parabola of L.R. $\frac{2v^2\cos^2\alpha}{g}$.

9. Two equal smooth spheres of radius r move with the same speed in opposite directions in parallel lines at a distance

c apart; prove that the motion of each deviates on impact through a right angle if $c^2(1+e)=4er^2$.

10. A smooth billiard ball strikes another of equal size which is at rest, and the direction of the centre of the former makes just before impact an angle of 30° with the line joining the centres of the two balls; find the tangent of the angle through which its direction of motion is deflected by the impact, e being 0·4 and the motion being supposed to take place on a smooth billiard table.

CHAPTER XVII.

THIRD LAW OF ROTATION. CONSERVATION OF ANGULAR MOMENTUM.

NEWTON'S Third Law of Motion states that action and reaction are equal and opposite, and from the Impulse equation $\int \mathrm{P}\,dt =$ change of momentum we deduce the principle of the Conservation of Linear Momentum.

Corresponding to this we have the Third Law of Rotation, which states that "When one body A exerts a torque on another body B, then an equal and opposite torque is exerted by B on A."

From the equation $\int \mathrm{C}\,dt =$ change of angular momentum (see p. 247), we deduce the principle of Conservation of Angular Momentum.

Suppose a disc A to be rotating with angular velocity Ω about a fixed axis, and that a second disc B, at rest, is mounted on the same axis. If B is slid along the axis until a projecting point begins to rub against A, a frictional force will be set up which will have the same moment about the axis for both bodies and will produce a torque tending to check A while at the same time setting B in motion.

If C be the torque brought into play and t the time during

which it acts, we have for the body A, whose moment of inertia is $\mathrm{W}k^2$,

$$\int \mathrm{C}\,dt = \mathrm{W}k^2 \frac{\Omega}{g} - \mathrm{W}k^2 \frac{\omega}{g},$$

if Ω is its initial and ω its final angular velocity about the axis.

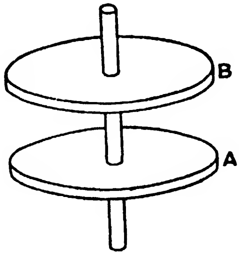

FIG. 176.

At the end of time t both bodies will have the same angular velocity, so that if the moment of inertia of B is $\mathrm{W}_1 k_1^2$, we have

$$\int \mathrm{C}\,dt = \mathrm{W}_1 k_1^2 \frac{\omega}{g}.$$

Hence $$\mathrm{W}k^2 \frac{\Omega}{g} - \mathrm{W}k^2 \frac{\omega}{g} = \mathrm{W}_1 k_1^2 \frac{\omega}{g},$$

i.e. $$\mathrm{W}k^2 \frac{\Omega}{g} = \mathrm{W}k^2 \frac{\omega}{g} + \mathrm{W}_1 k_1^2 \frac{\omega}{g},$$

which is equivalent to the statement that the angular momentum about a given axis is unaltered by the mutual action of the bodies.

Experiment. Stand on a turntable or on a music stool with revolving seat and hold the arms straight out in front. Then rotate one arm horizontally to the right; it will be found that the body turns slightly to the left. The angular momentum of the whole body was zero before the arm was moved, and remains zero since the angular momentum

of the arm is equal and opposite to the angular momentum of the rest of the body.

Extend the arms sideways and have the turntable set in motion. If now the arms are dropped to the side, the table will rotate more rapidly, since the radius of gyration of the body is reduced by bringing in the arms, and the angular velocity will increase in order to keep the angular momentum unaltered.

Angular Momentum and Moment of Momentum.

Let δw_1 be the weight of a small particle at P of a rigid body which is rotating about an axis at O with angular velocity ω.

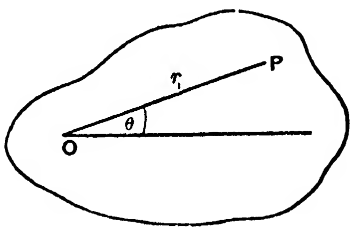

FIG. 177.

If OP $= r_1$ the velocity of the point P will be $r_1 \frac{d\theta}{dt}$, and the momentum of the particle will be

$$\delta w_1 \frac{r_1 \frac{d\theta}{dt}}{g}.$$

The moment of this momentum about O will be

$$\delta w_1 \frac{r_1^2 \frac{d\theta}{dt}}{g},$$

and the moment of momentum of the whole body about the axis will be

$$(\Sigma \delta w_1 r_1^2) \frac{\frac{d\theta}{dt}}{g} = Wk^2 \frac{\omega}{g},$$

where Wk^2 is its M.I. about the given axis. This is the expression which has been called the Angular Momentum of the body; hence the total moment of momentum of a rigid body about an axis is the same as its Angular Momentum about that axis.

Impulsive Torque.

If the change in angular momentum is produced by a blow P acting for a very short time t, the impulse of the blow will be Pt and the moment of the impulse will be Prt, where r is the perpendicular distance of the line of action of the force from the axis. If $Pr = C$, the torque, then Ct measures the moment of the impulse and is called the impulsive torque. Hence the impulsive torque about the axis equals the change in angular momentum about that axis. See p. 298.

EXAMPLE 1. A shaft A, whose M.I. is 40 lbs.-ft.^{2}, revolving at the rate of 400 revs./min., is fitted with a cog wheel with 20 teeth which engages in the cogs of a wheel having 60 teeth, fitted to another shaft B, whose M.I. is 20 lbs.-ft.^{2} making 100 revs. per min. Find the number of revolutions the shafts make after engaging.

Let r_1 be the radius of A and r_2 the radius of B.

If x is the distance between the cogs, then $20x = 2\pi r_1$ and $60x = 2\pi r_2$.

$$\therefore \frac{r_1}{r_2} = \frac{1}{3}.$$

Let n be the number of revolutions per min. made by A after engaging with B; then $\frac{20}{60} n$ is the number made by B, and if ω is the angular velocity of A, $\omega = \frac{n 2\pi}{60}$.

Let P be the force at the point of contact of the cogs at any instant; then the impulsive torque on A

$$=\int \mathrm{P}\,dt\,.\,r_1.$$

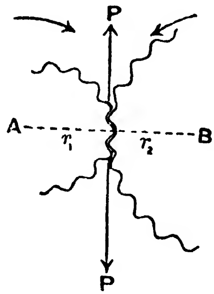

FIG. 178.

$$\therefore\ r_1\int \mathrm{P}\,dt=\text{change of angular momentum}$$

$$=\frac{40}{g}\left[400\frac{2\pi}{60}-\omega\right],$$

and
$$r_2\int \mathrm{P}\,dt=\frac{20}{g}\left[\frac{20}{60}\omega-100\frac{2\pi}{60}\right].$$

$$\therefore\ \frac{40\left[400\,.\,\frac{2\pi}{60}-n\,.\,\frac{2\pi}{60}\right]}{r_1}=\frac{20\left[\frac{1}{3}\,.\,n\frac{2\pi}{60}-100\frac{2\pi}{60}\right]}{r_2}$$

$$\therefore\ n\left[\frac{20}{9}+40\right]=400(40)+\frac{100(20)}{3}.$$

$\therefore\ n=395$, and the number of revolutions made by B

$$=\frac{n}{3}=132.$$

EXAMPLE 2. A boy in a swing is let go from A and crouches down until he reaches the lowest point B; he then stands up until he reaches C. If his radius of gyration about the axis C is k_1 from A to B and k_2 from B to C, and the distances of his

C. of G. from the axis are h_1 and h_2 respectively, find the angle BOC if AOB is α.

If ω is his angular velocity about O when at B, his K.E. at B $= Wk_1^2 \frac{\omega^2}{2g}$, which equals the work done by W

$$= Wh_1(1 - \cos \alpha) = 2Wh_1 \sin^2 \frac{\alpha}{2}.$$

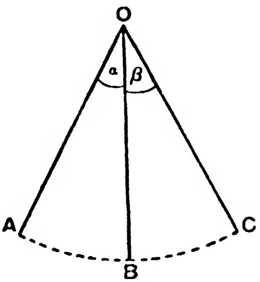

FIG. 179.

If ω_1 is his new angular velocity at B when standing up, we have $Wk_1^2 \frac{\omega}{g} = Wk_2^2 \frac{\omega'}{g}$, since his angular momentum about O is unaltered. Also by the principle of energy,

$$Wk_2^2 \frac{\omega_1^2}{2g} = Wh_2(1 - \cos \beta) = 2Wh_2 \sin^2 \frac{\beta}{2}.$$

$$\therefore \sin^2 \frac{\beta}{2} = \frac{h_1 k_1^2}{h_2 k_2^2} \sin^2 \frac{\alpha}{2}.$$

EXAMPLES XL.

1. Find the angular momentum of a uniform flywheel weighing 20 lbs., whose radius is 1 ft., when rotating about an axis through its centre at 5 radians per sec.

2. A thin uniform rod 4 ft. long, weighing 2 lbs., can rotate in a vertical plane about an axis at one end. It is held horizontally and allowed to fall. Find its angular momentum about the axis as it passes through the vertical position.

3. Two wheels on spindles in fixed bearings suddenly engage so that their angular velocities become inversely proportional to their radii and turn in opposite directions. One wheel of M.I. I_1 and radius a has an angular velocity ω_1 initially, and the other wheel of M.I. I_2 and radius b is initially at rest. Find the new angular velocity of each

4. A weight of 10 lbs. is revolving in a horizontal circle of 2 ft. radius at the rate of 2 turns per sec. It is brought in by a pull towards the centre until it describes a circle of radius 1 ft. What is now its angular velocity and its K.E. ? Find the work done in bringing it in.

5. A boy of weight W standing on a turntable rotating with angular velocity ω_1 stretches out his arms, thus increasing his radius of gyration from k_1 to k_2. Show that

$$\frac{\text{Final K.E.}}{\text{Initial K.E.}} = \frac{\text{Initial Moment of Inertia}}{\text{Final Moment of Inertia}}.$$

6. A flywheel of M.I. I is keyed to an axle of radius r and is rotating with angular velocity ω. As it rotates it winds up on the axle a light inextensible string which is attached to a mass M resting on the ground below. Show that at the instant the string becomes tight the angular velocity of the flywheel is reduced in the ratio $\frac{I}{I+Mr^2}$, and that the K.E. of the system is reduced in the same ratio.

7. A wheel weighing 50 lbs., radius 2 ft., whose mass may all be supposed collected at the rim, is rotating at the rate of 10 turns a second when a brake is applied for 6 secs., which reduces the speed to 1 turn per second. If the coefficient of friction between the wheel and the brake is 0·2, find the pressure of the brake on the wheels.

8. A uniform flywheel of mass 20 lbs. and radius 1 ft. has an axle of diameter 4″, whose mass may be neglected. When the wheel is making 5 turns per sec. a string round the axle jerks a weight of 50 lbs. from the floor. With what velocity will the weight leave the floor, and what will then be the angular velocity of the flywheel ?

9. A uniform flywheel weighing 5 lbs., of radius 8″, is rotating about a vertical axis and is making 4 turns per sec. when a piece of putty weighing 8 ozs. is dropped gently on it at a point 6″ from the centre. Find the number of turns per sec. the wheel then makes.

10. A cubical block of wood of weight W_1, whose radius of gyration about its C. of G. is k, is suspended by cords so that it can rotate in a vertical plane about an axis at O whose distance from G is l. A bullet of weight W_2 is fired with velocity v into the block in a line with G. Find the initial angular velocity of the block about O. If it rises a vertical height h, prove that its time of oscillation, assuming the angle through which it turns to be small, is $\sqrt{\frac{2l}{h}}\,\frac{\pi v}{g}\,\frac{W_2}{W_1+W_2}$.

11. A flywheel making 2 revolutions per second is braked by a belt round its circumference, and is cooled by water supplied to it at a steady rate of 12 gallons per min., which is carried round in a circle of 3 ft. radius. If the velocity of supply is negligible, find the retarding couple the water produces.

12. A rectangular board ABCD able to turn in a vertical plane about a horizontal axis at E, the mid point of the top edge AB, is given a sudden blow along DC. If the board turns through 60° before swinging back, find the impulse of the blow, given AB = 1 ft., AD = 3 ft., weight of board 6 lbs.

13. A solid rectangular body 4 ft. high, weighing 50 lbs., rests with its base, which is a square of side 3 ft., on a railway truck with one edge AB at right angles to the direction of motion. If the truck when moving at the rate of 10 ft./sec. suddenly stops, find the angular velocity about AB with which the body begins to turn, assuming that it does not slip.

14. A solid cylinder weighing 120 lbs., whose radius is 6″, is pressed down when at rest by a lever carrying its bearings so that its curved surface comes in contact with the surface of another cylinder weighing 400 lbs., whose radius is 8″ when the latter is rotating at a speed of 10 revolutions per sec. Find the number of revolutions per second of each cylinder when the speed of both becomes constant. If the pressure between

the cylinders is 150 lbs. and the time taken to acquire the steady speed is $1\frac{1}{2}$ secs., find the coefficient of friction.

15. Two solid cylinders each weighing 60 lbs., of radius 6 inches, are rotating in the same direction with their axes horizontal and parallel. A long bar weighing 100 lbs. is placed gently on the top of the cylinders at right angles to their axes, when one of the cylinders is making 2 revolutions per sec. and the other is making 3. Find the number of revolutions they make when the speed becomes steady and also the velocity of the bar.

16. Three equal particles of weight W are fastened to the corners of an equilateral triangular area ABC, whose mass is negligible, and the system is rotating about A with angular velocity ω. Find the moment of its momentum about the mid point of AB, and show that if A is released and the mid point of AB is suddenly fixed, then the angular velocity is unaltered. (C.U.)

CHAPTER XVIII.

ROTATION. INSTANTANEOUS CENTRE. MOTION RELATIVE TO THE C. OF G. GENERAL EQUATIONS OF MOTION FOR A RIGID BODY.

Angular Velocity of a Moving Body.

Suppose the body to be moving in one plane but not rotating about a fixed axis. Take any line BB′ in the plane and any fixed line AB in the body. Let the angle

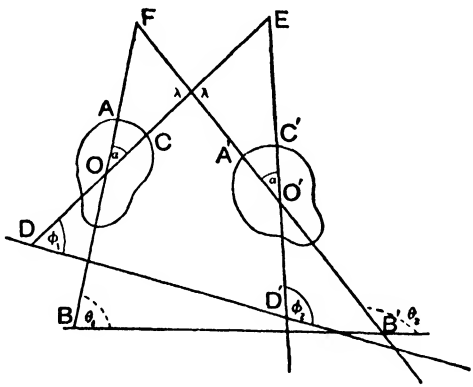

FIG. 180.

between these lines in the two positions of the body be θ_1 and θ_2 at an interval of time δt. Then $\dfrac{\theta_2 - \theta_1}{\delta t}$ measures the average angular velocity of the body.

If we take any other fixed line CD in the body and any other line DD′ in the plane, then, if the angles between these lines at the interval of time δt are φ_1 and φ_2, the average angular velocity would be $\frac{\varphi_2 - \varphi_1}{\delta t}$.

But $\varphi_2 - \varphi_1 = \text{DED}'$ (see Fig. 180) and $\theta_2 - \theta_1 = \text{BFB}'$, but since $\text{FOE} = \text{FO}'\text{E} = a$,

$$\therefore \ \text{DED}' = \pi - (\lambda + a) = \text{BFB}'.$$

$$\therefore \ \frac{\varphi_2 - \varphi_1}{\delta t} = \frac{\theta_2 - \theta_1}{\delta t}.$$

Hence it is immaterial which line we select either in the plane or in the body in order to measure by the rotation of the latter the angular velocity of the moving body.

Velocity of a Point on a Rolling Wheel.

Experiment. Place two pencils parallel to one another with a board over them, as in Fig. 181. Push the board along

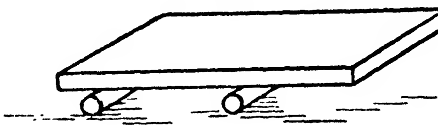

FIG. 181.

so that the pencils roll on the table without slipping. Note the distance travelled by the board while the pencils move two inches—it will be found to have moved four inches.

The board has the same velocity as the part of the pencils with which it is in contact, so that at any instant it appears that the velocity of the highest point of a rolling circle is twice the velocity of the centre.

We shall obtain a clear idea of the motion of a circle rolling along a line AX if we suppose the circle provided

with a series of spikes round its circumference. When the point A reaches the line, the whole circle will rotate about the point A until B reaches the ground; during this time the

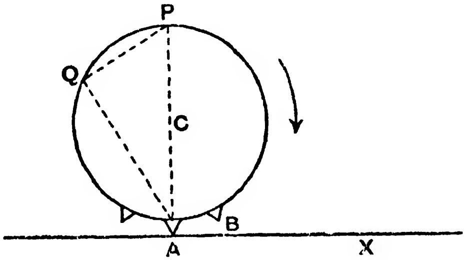

FIG. 182.

point A is stationary, and the point P at the other end of the diameter will move twice as fast as the centre C. Any other point Q will move at right angles to AQ, and QP which is at right angles to AQ, will be the direction of the tangent to Q's path.

These results may be proved by calculating the velocity of any point on the rolling circle, as follows:

Let the velocity of the centre be v. While the circle

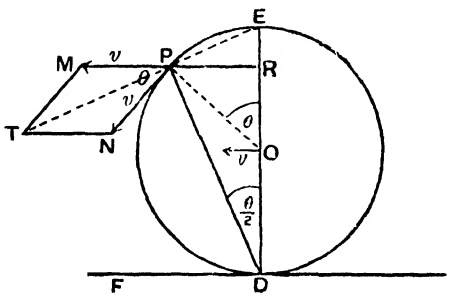

FIG. 183.

makes a complete revolution, a point P on the circumference rotates once round O, so that the whole circle travels

a distance $2\pi r$ along DF while P travels $2\pi r$ round O; the velocity of P relative to O is therefore also v. But P is also moving horizontally with the same velocity as the whole circle. $\therefore$ P's velocity in space is the resultant of v along PM and v along PN the tangent to the circle at P. $\therefore$ the resultant velocity PT bisects MPN. $\therefore$ if MPN $=\theta$, MP makes an angle $\frac{\theta}{2}$ with PT.

Join P to E, the extremity of the diameter through D and O; then POE $=\theta$.

$$\therefore \text{ OEP} = 90^\circ - \frac{\theta}{2}. \qquad \therefore \text{ RPE} = \frac{\theta}{2}.$$

$\therefore$ TPE is a straight line and PT is at right angles to PD.

$\therefore$ P is instantaneously moving round the point D with velocity PT $= 2v \cos \frac{\theta}{2}$.

For the point E, $\theta = 0^\circ$. $\therefore$ the velocity of E $= 2v$, and for D, $\theta = 180^\circ$. $\therefore$ the velocity of D is zero, and D is instantaneously at rest while every other point in the circle is moving round it.

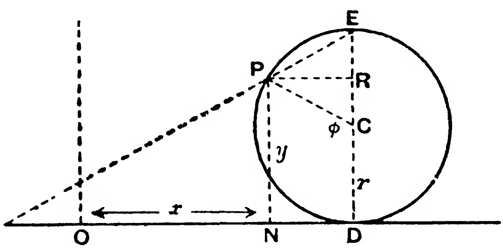

FIG. 184.

The coordinates of P when the circle has turned through an angle φ starting from a point O can be obtained thus:

Since P originally was at O, OD = arc PD = $r\varphi$.

$$\therefore\ x = \text{ON} = \text{OD} - \text{PR} = r\varphi - r\sin\varphi.$$

$$y = \text{PN} = \text{DC} + \text{CR} = r - r\cos\varphi.$$

To find the velocity of P by differentiating, we have

$$\dot{x} = r\dot{\phi} - r\cos\phi\dot{\phi}.$$

$$\dot{y} = r\sin\phi\dot{\phi}.$$

$$\therefore\ v^2 = \dot{x}^2 + \dot{y}^2 = r^2(1-\cos\varphi)^2\omega^2 + r^2\sin^2\varphi\,\omega^2,$$

where $\dot{\phi} = \omega$, the angular velocity of the circle.

$$\therefore\ v^2 = 2r^2\omega^2(1-\cos\varphi) = 4r^2\omega^2\sin^2\frac{\varphi}{2}.$$

$$\therefore\ v = 2r\omega\sin\frac{\varphi}{2}.$$

The direction of P's motion is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{r\sin\varphi}{r(1-\cos\varphi)} = \frac{2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}{2\sin^2\frac{\varphi}{2}} = \cot\frac{\varphi}{2} = \tan(90^\circ - \text{PEC})$$

$$= \tan \text{EPR}.$$

P therefore is moving in the direction PE.

Centre of Rotation.

To show that a body moving in a plane may be brought from any one position to any other position by rotating it through a certain angle about some point.

If we change the position of any given line in the body from the old position to the new, we shall have made the required change in the whole of the body.

Suppose A and B to be the original positions of the ends of the line AB, and A′B′ to be their new positions.

Bisect AA′ and BB′ at right angles, and let the bisectors meet at O.

Since AO = A′O, BO = B′O, AB = A′B′,

$$\therefore\ A\hat{O}B = A'OB'.$$

To each add BOA′; then AOA′ = BOB′. ∴ if we rotate the body round O so that A comes to A′, the same rotation will bring B to B.′ O is the centre of rotation.

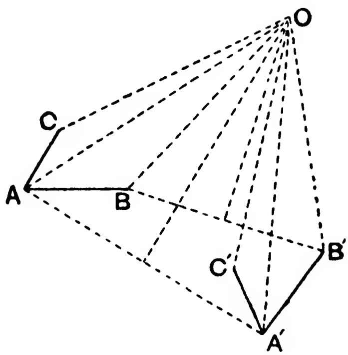

FIG. 185.

Any other point C in the body will of course describe a circle round O, and if C′ is its new position OC = OC′, for since OAB and OA′B′ are congruent triangles,

$$\therefore\ OAB = OA'B'.$$

Also CAB = C′A′B′. ∴ OAC = OA′C′.

And OA = OA′, AC = A′C′. ∴ OC = OC′.

If the motion is purely translational, the line AB moves parallel to itself, and O, the centre of rotation, is at an infinite distance.

Displacement Theorem.

Any displacement of a rigid body can be effected by a motion of translation of any one point together with a rotation about that point equal to the rotation about the

centre of rotation which would be required to bring the body into the new position.

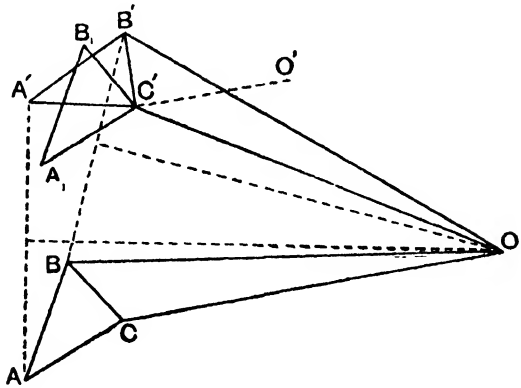

FIG. 186.

Let any line AB in the body be displaced into the position A′B′, and find O the centre of rotation as before.

Let C be any other point, and let C′ be the new position of C.

Draw $C'A_1B_1$ with its sides parallel to the sides of CAB. Then the displacement of the body can be effected by a translation of C to C′ together with a rotation $B_1C'B'$ about C′. Draw C′O′ parallel to CO. $OC = OC'$, $OB = OB'$, $CB = C'B'$.

$$\therefore\ O\hat{C}B = O\hat{C}'B'.$$

But $OCB = O'C'B_1$. $\quad\therefore\ O\hat{C}'B' = O'\hat{C}'B_1$.

$$\therefore\ OC'O' = B_1C'B' \text{ and } O'C'O = C'OC.$$

$$\therefore\ C'OC = B_1C'B'.$$

∴ the rotation $B_1C'B'$ about C′ is equal to the rotation COC′ about O that would have brought AB to A′B′. The point C is usually the C. of G. of the body, as will appear subsequently.

Instantaneous Centre of Rotation.

We have seen that in the case of the wheel rolling along the road the point of contact between the wheel and road is momentarily at rest, and every particle of the wheel may be considered for the instant to be moving about it. A new point then comes in contact with the road, and the motion may again be considered as a rotation about this point. Such a point is called the instantaneous centre.

Any very small motion of a body, except one of pure translation, may be considered to be due to a rotation about an instantaneous centre.

Such a point may often be detected from the following considerations:

(1) If we know that any point in the body is momentarily at rest, that point must be the instantaneous centre.

(2) If the direction of motion of any point in a body is known, the instantaneous centre must lie on a line through the point at right angles to its direction of motion.

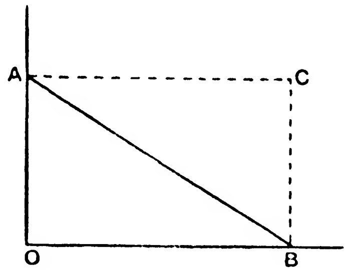

FIG. 187.

E.g. if a rod AB is sliding between two fixed rods OA, OB, the point A is moving along AO, and is therefore momentarily rotating about a point in AC at right angles to AO.

Similarly B must be rotating about a point in BC. $\therefore$ C is the instantaneous centre.

Suppose a very small displacement to take a point A to A′ while B moves to B′. Then AOA′ = BOB′, where O is the instantaneous centre.

$$\therefore \frac{AA'}{AO} = \frac{BB'}{BO}.$$

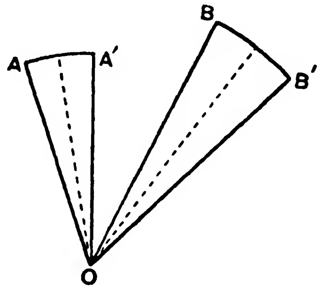

FIG. 186.

But the time taken is the same in both cases. $\therefore$ AA′ $= ut$ and BB′ $= vt$ when u is the velocity of A and v of B.

$$\therefore \frac{u}{AO} = \frac{v}{BO}.$$

$\therefore$ the velocity of the points are proportional to their distances from the instantaneous centre.

In the case of the rolling wheel we have (Fig. 183)

$$\frac{\text{vel. of P}}{\text{vel. of E}} = \frac{2v\cos\frac{\theta}{2}}{2v} = \cos\frac{\theta}{2} = \frac{PD}{ED}.$$

Crank Motion.

A crank OP rotates about a fixed centre O while the end B of the connecting rod PB is constrained to move along OB. If the angular velocity of P is ω, determine the velocity of B.

Since B is moving along BO, the instantaneous centre of the rod PB must be on the line BD at right angles to OB.

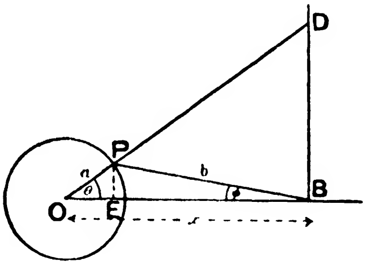

FIG. 189.

Since P is moving at right angles to OP, the instantaneous centre must be on OP. $\therefore$ it is at D.

Now $$\frac{\text{vel. of P}}{\text{vel. of B}} = \frac{\text{PD}}{\text{BD}} = \frac{\sin \text{PBD}}{\sin \text{DPB}} = \frac{\cos\varphi}{\sin(\theta+\varphi)}.$$

$$\therefore \text{ vel. of B} = \text{vel. of P} \times \frac{\sin(\theta+\varphi)}{\cos\varphi} = a\frac{d\theta}{dt}\frac{\sin(\theta+\varphi)}{\cos\varphi}.$$

Alternative Proof by the Calculus.

Let OB $= x$; then $x = a\cos\theta + b\cos\varphi$ (Fig. 189).

$$\therefore \frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt} - b\sin\varphi\frac{d\varphi}{dt}.$$

But $a\sin\theta =$ PE $= b\sin\varphi$.

$$\therefore a\cos\theta\frac{d\theta}{dt} = b\cos\varphi\frac{d\varphi}{dt}.$$

$$\therefore \frac{dx}{dt} = -a\sin\theta\frac{d\theta}{dt} - b\sin\varphi\left(\frac{a\cos\theta}{b\cos\varphi}\frac{d\theta}{dt}\right)$$

$$= -a\frac{(\sin\theta\cos\varphi + \sin\varphi\cos\theta)}{\cos\varphi}\frac{d\theta}{dt} = -a\frac{\sin(\theta+\varphi)}{\cos\varphi}\frac{d\theta}{dt},$$

the negative sign showing that x is decreasing when θ is increasing.

EXAMPLE 1. A cogwheel A of radius a turns freely round its centre C which is carried by a bar OC, and also is in gear with a fixed cog wheel B of radius b. If the bar OC rotates round O with angular velocity ω, find the angular velocity Ω of the wheel A.

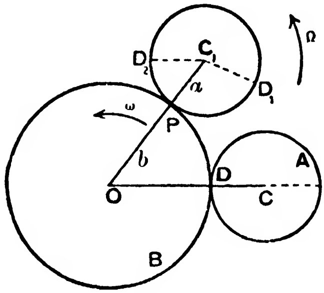

FIG 190.

When C moves to the position C_1, the point of contact D will move to D_1 and the radius CD will have turned through the angle, $D_2C_1D_1$, where D_2C_1 is parallel to CD.

The point P in contact with the stationary wheel B is instantaneously at rest.

Now the velocity of P = velocity of C_1 − velocity of P relatively to C_1 (both at right angles to OC_1).

$$\therefore\ 0=(a+b)\omega-a\Omega. \quad \therefore\ \Omega=\frac{a+b}{a}\cdot\omega.$$

Prove this result by using the fact that arc PD_1 = arc PD, and that $D_2\hat{C}_1D_1=D_2\hat{C}_1O+O\hat{C}_1D_1$.

EXAMPLES XLI.

1. The wheels of a bicycle are 30″ in diameter, and the gear ratio between the crank axle and wheel axle is $2\frac{1}{2}$; the length of the crank is 8″. Find the velocity of the end of the crank and the magnitude and direction of its acceleration when at its highest point, the bicycle travelling at the rate of 30 ft./sec.

2. The ends of a rod AB move along two fixed lines OA, OB, at right angles to one another. Find the ratio of the velocities

of the ends, and prove that the components of the velocities along AB are the same.

3. A straight line AB is moving in any manner in a plane; prove that the velocity of B relative to A is at right angles to AB.

4. Sketch the curve described in space by a point P on a circle which rolls along a straight line AB.

5. In Fig. 189 draw ON perpendicular to OB to meet BP produced at N, and show that velocity of P : velocity of B : velocity of B relative to P = OP : ON : PN.

6. In Fig. 189, if PB is produced, and a groove cut in it so that a knob B fixed on OB slides in the groove, prove that the velocity of the part of PB which is touching B is $V \sin(\theta+\varphi)$, where V is the velocity of P.

7. AB is a straight rod of length 3 ft. At a given instant the end A is moving with a velocity of 10 ft./sec. at an angle of 45° to AB, and the end B is moving at an angle of 120° to BA on the opposite side of AB. Find the velocity of B and of the mid point of the rod. What is the angular velocity of the rod?

8. A ladder AB, with one end on a horizontal floor OA, and the other end B on a vertical wall OB, is sliding down. Show that when the ladder makes an angle θ with the floor the direction of motion of its mid point makes an angle θ with the wall. If A is moving with uniform velocity, prove that the angular velocity of the ladder is inversely proportional to $\sin\theta$.

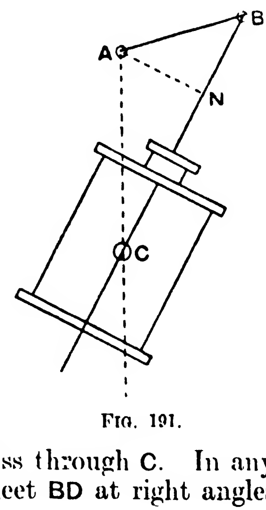

Fig. 191.

9. Fig. 191 represents an oscillating cylinder which rotates about an axis at C. The crank AB rotates with angular velocity ω about A; find the angular velocity of the cylinder.

10. A and C are given points in a plane, in which a bar AB is turning about A with angular velocity ω. AB is jointed at B to a bar BD which is constrained to pass through C. In any position of the linkwork draw AE to meet BD at right angles

in E; draw EF parallel to AB to meet AC in F; let ω' be the angular velocity of BD. Prove that the velocity of the point in BD which is passing through C is $\omega \cdot$ AE; find the instantaneous centre for the rod BD, and prove $\omega' = \frac{AF}{AC} \cdot \omega$.

11. Two concentric cog wheels A and B, of radius a and b, rotate in the same sense with angular velocities ω_1 and ω_2 round their centre O. A third cog wheel C gears with teeth on the inner edge of A and the outer edge of B, and the centre of this cog describes a circle round O with angular velocity ω_3. If Ω is the angular velocity of the pinion wheel C, prove that

(i) $$a\omega_1 = \frac{a+b}{2}\omega_3 - \frac{a-b}{2}\Omega.$$

(ii) $$b\omega_2 = \frac{a+b}{2}\omega_3 + \frac{a-b}{2}\Omega.$$

Find ω_2 if A is fixed and ω_1 if B is fixed.

12. O is the centre of a circle of radius 1″ and A is a point 3·5″ from O. Draw OAQ = 50° and make AQ = 2″. Join Q to a point P on the circle cutting OA at E. AQ represents a beam pivoted at A and QP is a connecting rod to a crank OP rotating about a fixed centre O. Draw OF parallel to AQ cutting QP produced at F. Prove

$$\frac{\text{vel. of Q}}{\text{vel. of P}} = \frac{\text{OF}}{\text{OP}} = \frac{\text{AQ}}{\text{OP}} \times \frac{\text{OE}}{\text{AE}}.$$

If QP = 2″, find graphically the ratio of these velocities in the given position.

13. Two cranks AB and CD are connected by a coupler BD. AB = 1 ft., CD = $1\frac{1}{2}$ ft., BD = $2\frac{1}{2}$ ft., AC = 3 ft. When BAC = 45°, the angular velocity of AB = 20 radians per sec. What is the angular velocity of CD and of BD? What couple applied to CD will balance a couple of 15 lbs.-ft. applied to AB?

Motion relative to the Centre of Gravity.

We have seen (p. 326) that any displacement of a rigid body may be effected by a motion of translation of any one point combined with a rotation about that point.

The point generally selected is the C. of G. of the body, and the reason for this choice will appear from the following experiments.

Experiment 1. Place a book on a table and push it along the table with a stick held horizontally. If the line of the stick is such that it passes through the C. of G. of the book, the book will move without turning; for any other direction the book will turn.

Experiment 2. Hold a bar weighted at one end vertically and let it fall. While the bar is falling strike it sharply in a horizontal direction with a stick, and repeat the experiment until a point on the bar is found such that, when struck at that point, the bar does not turn. It will be found that in this case the direction of the blow passes through the C. of G. of the body.

Experiment 3. Suspend horizontally by a string a heavy bar weighted at one end. Place a vertical metre measure behind the string and strike the bar sharply vertically upwards with a stick. Note the motion of the C. of G. of the bar. It will be found that the C. of G. rises vertically; and that the bar turns round it unless the blow is delivered at the C. of G.

Motion of the Centre of Gravity of a System of Particles.

Let x_1, y_1, z_1; x_2, y_2, z_2, etc., be the coordinates referred to the axes OX, OY, OZ of a system of particles whose weights are w_1, w_2, etc.

Then by taking moments about the axes in turn, the coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the C. of G. of the particles are given by

$$\Sigma w(\bar{x}) = \Sigma wx\,;\ \Sigma w(\bar{y}) = \Sigma wy\,;\ \Sigma w(\bar{z}) = \Sigma wz.$$

Differentiating with respect to t, we have if W is the total weight

$$\mathrm{W}\frac{d\bar{x}}{dt}=w_1\frac{dx_1}{dt}+w_2\frac{dx_2}{dt}+\ldots$$

and similar results for

$$\mathrm{W}\frac{d\bar{y}}{dt} \quad \text{and} \quad \mathrm{W}\frac{d\bar{z}}{dt}.$$

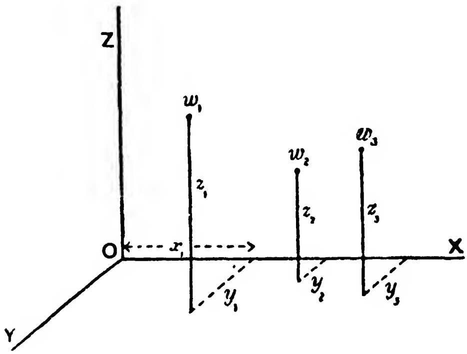

FIG. 192.

(1) Hence $$\mathrm{W}\frac{\dot{\bar{x}}}{g}=w_1\frac{\dot{x}_1}{g}+w_2\frac{\dot{x}_2}{g}+\ldots$$

and therefore the total momentum of the system in any direction equals the momentum of the whole mass collected at the C. of G.

Differentiating (i) with respect to t, we have

$$\mathrm{W}\frac{\ddot{\bar{x}}}{g}=w_1\frac{\ddot{x}_1}{g}+w_1\frac{\ddot{x}_2}{g}+\ldots$$

which gives the acceleration of the C. of G. if the accelerations of the various particles are known.

If these particles compose a rigid body, $w_1\frac{\ddot{x}_1}{g}$, etc., will represent the components, parallel to OX, of the resultant

forces acting on the particles due to the action of the neighbouring particles and to the external forces. When these forces are added together for the whole body the internal reactions will disappear, since to each force there is an equal and opposite reaction, and it follows that "The acceleration of the C. of G. in the direction OX of a rigid body is given by the equation $W\frac{\ddot{x}}{g}$ = sum of the resolved parts of the external forces in the direction OX.

Similar expressions give us the acceleration of the C. of G. in the directions OY and OZ.

Hence the acceleration of the C. of G. in any direction OX is given by

$$\frac{\text{Sum of resolved parts of the forces acting on the body in the direction OX}}{W} = \frac{\ddot{x}}{g}.$$

Kinetic Energy of a Body rotating about a Fixed Axis.

If the body is rotating with angular velocity ω about an axis at O then its K.E. is $Wk_1^2\frac{\omega^2}{2g}$ when k_1 is the radius of gyration about O. But $Wk_1^2 = Wk^2 + Wa^2$ when k is the radius of gyration about a parallel axis through G the C. of G. and a is the distance OG.

$$\therefore \text{ K.E.} = W(k^2+a^2)\frac{\omega^2}{2g} = Wk^2\frac{\omega^2}{2g} + W\frac{a^2\omega^2}{2g}$$
$$= Wk^2\frac{\omega^2}{2g} + W\frac{v^2}{2g},$$

where v is the velocity of G since the angular velocity of G relative to O is ω.

$\therefore$ K.E. of the rotating body equals the K.E. of the body with all its mass supposed collected at G together

with the K.E. of the body rotating about G with angular velocity ω.

Extension to the Case of a Body moving in any manner in two Dimensions.

Let the components of the velocity of the C. of G. parallel to the axes be u and v, and ω the angular velocity of the body.

Consider a particle at P of weight δw_1 whose distance from G is r_1, then P is moving with the velocity of G together with an angular velocity ω about G, *i.e.* with velocity $r_1 \frac{d\theta}{dt}$ at right angles to PG. $\therefore$ the components of its velocity in space will be $u - r_1 \frac{d\theta}{dt} \sin\theta$ parallel to OX, and

$$v + r_1 \frac{d\theta}{dt} \cos\theta \text{ parallel to OY.}$$

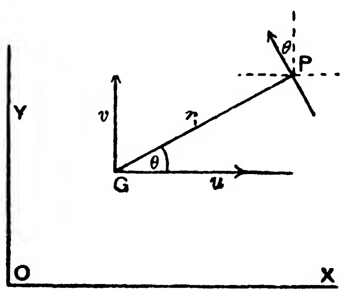

FIG. 193.

$$\therefore \text{ K.E. of } \delta w_1 = \frac{\delta w_1}{2g}[v^2 + 2vr_1\dot{\theta}\cos\theta + r_1{}^2\dot{\theta}^2\cos^2\theta + u^2 - 2ur_1\dot{\theta}\sin\theta + r_1{}^2\dot{\theta}^2\sin^2\theta],$$

and K.E. of the whole body

$$= \frac{1}{2g}\{\Sigma\delta w(u^2+v^2) + \Sigma\delta w r^2\dot{\theta}^2 + 2\dot{\theta}v\Sigma\delta w r\cos\theta - 2\dot{\theta}u\Sigma\delta w r\sin\theta\}.$$

But $\Sigma\delta wr\cos\theta = \Sigma\delta wx$ and $\Sigma\delta wr\sin\theta = \Sigma\delta wy$, where x and y are the coordinates of any point in the body with reference to parallel axes through G, but since G is the C. of G. of the body $\Sigma\delta wx = 0$, and $\Sigma\delta wy = 0$.

$\therefore$ K.E. of the whole body

$$= \Sigma\delta w\frac{(u^2+v^2)}{2g} + \Sigma(\delta wr^2)\frac{\dot{\theta}^2}{2g}$$

$$= \mathrm{W}\frac{(u^2+v^2)}{2g} + \mathrm{W}k^2\frac{\omega^2}{2g}$$

= K.E. of the whole mass moving with the velocity of the C. of G. + K.E. of the body rotating about the C. of G.

Angular Momentum of a Body rotating about a Fixed Axis.

If the body is rotating about an axis at O with angular velocity ω, its angular momentum about O is $\mathrm{W}(k^2+a^2)\frac{\omega}{g}$,

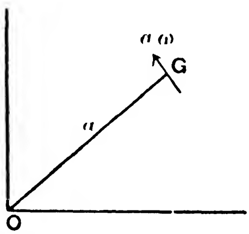

FIG. 194.

where k is the radius of gyration about a parallel axis through G. This expression may be written

$$\mathrm{W}k^2\frac{\omega}{g} + \mathrm{W}\frac{a^2\omega}{g}.$$

The velocity of the C. of G. is $a\omega$ since it is rotating about O with angular velocity ω. $\therefore$ the momentum of the

whole mass supposed collected at the C. of G. would be $W\dfrac{a\omega}{g}$, and the moment of its momentum about O would be $W\dfrac{a^2\omega}{g}$.

$\therefore$ The Angular Momentum of the body about O = Angular Momentum about G + the moment of momentum about O of the whole mass supposed collected at the C. of G.

Extension to the case of a Body moving in any manner in a Plane.

A proof similar to that on p. 336 will show that the Angular Momentum of a body about any axis equals the moment of the momentum of the mass collected at the C. of G. about that axis together with the angular momentum of the body about a parallel axis through its C. of G. If the C. of G. of a body of weight W is moving with a velocity whose components parallel to the axes are u and v, and if the body is also rotating with angular velocity ω, then the moment of its momentum about any point O

$$= W\frac{v}{g}\cdot\bar{x} - W\frac{u}{g}\bar{y} + Wk^2\frac{\omega}{g},$$

where $\bar{x}$ and $\bar{y}$ are the coordinates of the C. of G. referred to axes through the point O.

EXAMPLE. A uniform cylinder weighing 50 lbs. whose radius is $\frac{1}{2}$ foot, rolls along a horizontal plane with a velocity of 5 ft. per sec. Find its K.E.

A point on the circumference describes 5 ft. in one second relatively to the centre of the disc.

$$\therefore \text{ angular velocity of the cylinder} = \frac{5}{\frac{1}{2}} = 10 \text{ rads./sec.}$$

$$\text{K.E. of the cylinder} = 50(\tfrac{1}{2})^2\tfrac{1}{2}\cdot\tfrac{100}{64} + 50\frac{5^2}{64}.$$

It will be noticed that this is the same as finding the K.E. of the cylinder by considering that it is rotating at any instant about the point of contact with the ground, *i.e.* the Instantaneous Centre.

EXAMPLES XLII.

1. Two bodies of weight W_1 and W_2 are fastened to the ends of a string passing over a small pulley. Find the velocity and acceleration of their C. of G. when W_1 descends with velocity v.

2. A rope is coiled round a drum of radius 1 foot. If the drum rotates with angular velocity of 4 radians per sec., find the velocity of the C. of G. of the hanging part of the rope.

3. A body weighing 5 lbs. lying on a smooth table is connected by a light string to a body weighing 2 lbs. which hangs over the edge. Find the magnitude of the acceleration of the C. of G. of the bodies.

4. Two strings are fastened to two points A and B 5 feet apart in the same horizontal line. A weight C of 5 lbs. is fastened to the string AC, which is 1 ft. long, and a weight D of 10 lbs. to the string BD, 2 ft. long. Another string joins C and D so that AC makes 45° and BD 30° with the vertical. If C is displaced, AC being kept taut, sketch the locus of the C. of G. of C and D, and show that it is in its lowest position when C and D are in their position of stable equilibrium.

5. Two small spheres, one 3 times the weight of the other, lie on a horizontal plane and are fastened by an elastic string. The bodies are pulled 4 ft. apart and then released. What is the velocity of their C. of G. during motion? Where do they meet?

6. Show that when a hoop rolls in a vertical plane, one half of its K.E. is rotational.

7. The ends of a light elastic string are fastened to the ends of a rod AB 2 ft. long which weighs 2 lbs. A weight of 1 lb. is attached to the mid point of the string. The system is placed on a smooth horizontal table, and AB is held while the string is stretched until it forms an equilateral triangle with

AB. If the rod and the weight are simultaneously released, find how far the weight will travel before meeting the rod.

8. A cylinder weighing 5 lbs. whose length is 2 ft. and radius 1 ft., is placed on a rough inclined plane so that its length is at right angles to a line of greatest slope and then released. When its axis is moving at 5 ft./sec. find its K.E.

9. A ball starts to roll down an inclined plane at the same instant that another equal ball begins to fall vertically from the bottom of the plane to the ground. When the first ball reaches the bottom of the plane the second reaches the ground vertically below the first. What is the path of the C. of G. of the two balls during the motion ?

10. Two bodies of weight W_1 and W_2 are connected by a string passing over the ridge of a wedge of weight W which rests on a smooth plane. What is the acceleration of the centre of mass of the whole system when motion ensues ? If a_1 is the acceleration of W_1 relative to the face of the wedge whose slope is α, a_2 the corresponding acceleration of W_2 on face of slope β and a_3 the horizontal acceleration of the wedge, prove that $Wa_3 + W_1(a_3 - a_1 \cos \alpha) + W_2(a_3 - a_2 \cos \beta) = 0$.

11. A sphere weighing 10 lbs. whose radius is 8″ is rolling on the ground at 4 ft./sec. Find its K.E.

12. A heavy uniform chain AB of length 9 ft. is held in a vertical position. If the upper end A is released find the velocity of the C. of G. of the chain as A passes B without striking it. If B is released at the instant A passes it, find the subsequent acceleration of the C. of G. of the chain, and prove that the time now taken to straighten the chain is $\frac{3}{4}$ of the time taken by A to fall to B.

13. The two wheels of a bicycle weigh W_1, their diameter is d_1 and radius of gyration k_1 ; the crank axle weighs W_2, its R. of G. is k_2, and it is geared to d_2 inches, the chain weighs W_3, and it moves over a toothed wheel of diameter d_3 on the driving axle. If the total weight of the bicycle is W and it is moving with velocity v, prove that its K.E. =

$$\left[W + W_1\frac{4k_1^2}{d_1^2} + W_2\frac{4k_2^2}{d_2^2} + W_3\frac{d_3^2}{d_1^2}\right]\frac{v^2}{2g}.$$

Reaction at the Axis of the Compound Pendulum.

Resolve the reaction at O, the axis, into two components X along the rod and Y at right angles to it. When the angular velocity of the rod is $\dot{\theta}$ and the angular acceleration $\ddot{\theta}$ the C. of G. will have an acceleration $h\dot{\theta}^2$ towards O (since it is moving in a circle about O) and an acceleration $h\ddot{\theta}$ at right angles to OG.

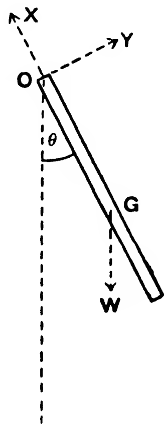

FIG. 195.

Since the acceleration of the C. of G. in any direction is given by the same equations as if the forces acting on the body were supposed to act at the C. of G. we have :

$$\frac{X - W\cos\theta}{W} = \frac{h\dot{\theta}^2}{g}$$

and

$$\frac{W\sin\theta - Y}{W} = \frac{h\ddot{\theta}}{g}.$$

These equations give X and Y; the angular velocity being determined at any instant, if the rod started from the position $\theta = \alpha$, by the energy equation

$$W(k^2 + h^2)\frac{\dot{\theta}^2}{2g} = Wh(\cos\theta - \cos\alpha),$$

where k is the radius of gyration of the rod about G.

Impulsive Reaction.

When the pendulum is at rest, suppose it given a horizontal blow whose impulse is **I** at a point A where AO $= a$. Let it start rotating with angular velocity ω, and let the impulse produced at O be Y. Now G begins to move with velocity $h\omega$ and the momentum of the system equals the momentum of the whole mass collected at G (Fig. 196).

$\therefore$ Resultant Impulse $= \mathbf{I} - \mathrm{Y} =$ change of Momentum

$$= \mathrm{W}\frac{h\omega}{g}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

If P is the force exerted at A, the torque on the rod will be $\mathrm{P}a = \mathrm{C}$, and if t is the duration of the blow $\mathrm{P}ta = \mathrm{C}t$, but $\mathrm{P}t = \mathbf{I}$, the impulse; $\therefore$ $\mathbf{I}a = \mathrm{C}t$, the impulsive torque.

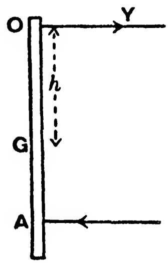

FIG. 196.

Since Impulsive torque $=$ Change of Angular Momentum about O, we have $\mathbf{I}a = \mathrm{W}(k^2 + h^2)\frac{\omega}{g}$ where k is the radius of gyration about G.

$$\therefore \; \mathrm{Y} = \mathbf{I} - \mathrm{W}h\frac{\omega}{g} \text{ from (i).}$$

$$= \mathbf{I} - \mathrm{W}h\frac{\mathbf{I}a}{\mathrm{W}(h^2 + k^2)}$$

$$= \mathbf{I}\left[1 - \frac{ha}{h^2 + k^2}\right].$$

$\mathrm{Y} = 0$ when $h^2 + k^2 = ha$, *i.e.* $k^2 = h(a - h)$, but $k^2 = h(l - h)$ (see p. 276), where l is the length of the simple equivalent pendulum. $\therefore$ when $\mathrm{Y} = 0$, $a = l$, *i.e.* there is no impulse produced at O if the blow is delivered at the centre of oscillation, which is, for this reason, also called the centre of percussion. The jar which is sometimes felt in the handle of a bat does not occur if the ball strikes the bat at the centre of percussion.

Equations of Motion of a Rigid Body rotating about an axis not fixed in space, but moving parallel to itself.

We have seen that any displacement of a rigid body in a plane can be effected by a motion of translation of any one point (say the C. of G.) together with a rotation about that point, and it has been proved on p. 334 that the acceleration of the C. of G. of the body is given by the same equations as would be obtained by assuming that the external forces were transferred to act at the C. of G.

It now remains to find the equations obtained by considering the rotation of the body.

Let the C. of G. of the body move with accelerations f_1 and f_2 respectively in the directions GX and GY, and suppose the forces in these directions acting on a particle δw_1 at P to be X and Y.

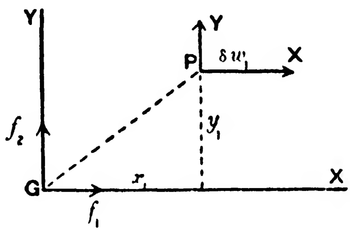

FIG. 197.

If the coordinates of P referred to GX and GY are x_1, y_1, we have, since the accelerations of P relative to G will be $\ddot{x}_1$ and $\ddot{y}_1$,

$$\frac{\mathrm{X}}{\delta w_1}=\frac{f_1+\ddot{x}_1}{g} \quad \text{and} \quad \frac{\mathrm{Y}}{\delta w_1}=\frac{f_2+\ddot{y}_1}{g}. \qquad \text{...........(i)}$$

Taking moments about G we have

$$\mathrm{Y}x_1-\mathrm{X}y_1=\frac{\delta w_1}{g}[(f_2+\ddot{y}_1)x_1-(f_1+\ddot{x}_1)y_1], \text{ from (i).}$$

Assuming that the internal reactions cancel one another when the forces acting on every particle are considered together, we shall be left with only the sum of the moments of the external forces.

$\therefore$ Sum of moments of external forces about G = L (say)

$$\frac{f_2\Sigma\delta w_1 x_1 - f_1\Sigma\delta w_1 y_1 + \Sigma\delta w_1(x_1\ddot{y}_1 - y_1\ddot{x}_1)}{g}.$$

Since the coordinates of G are 0, 0, we have

$$\Sigma\delta w_1 x_1 = 0; \quad \Sigma\delta w_1 y_1 = 0;$$

$$\therefore \text{L} = \frac{\Sigma\delta w_1(x_1\ddot{y}_1 - y_1\ddot{x}_1)}{g}.$$

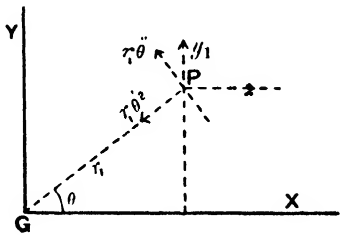

FIG. 198.

Let the polar coordinates of P be r_1, θ; since P is rotating about G, its acceleration at right angles to GP is $r_1\ddot{\theta}$ and along PG it is $r_1\left|\frac{d\theta}{dt}\right|^2$.

Resolving these parallel to GX and GY we have

$$\ddot{x}_1 = -r_1\dot{\theta}^2\cos\theta - r_1\ddot{\theta}\sin\theta, \quad \ddot{y}_1 = r_1\ddot{\theta}\cos\theta - r_1\dot{\theta}^2\sin\theta;$$

$$\therefore x_1\ddot{y}_1 - y_1\ddot{x}_1 = r_1\cos\theta(r_1\ddot{\theta}\cos\theta - r_1\dot{\theta}^2\sin\theta)$$
$$- r_1\sin\theta(-r_1\dot{\theta}^2\cos\theta - r_1\ddot{\theta}\sin\theta) = r_1^2\ddot{\theta};$$

$$\therefore \text{L} = \frac{\ddot{\theta}\Sigma\delta w r_1^2}{g} = \text{W}k^2\frac{\ddot{\theta}}{g}$$

where Wk^2 is the moment of inertia of the body about an axis through G.

This equation is the same as the one obtained when the body rotated about an axis through G, which was fixed in space. The equations for a body rotating about an axis moving parallel to itself are therefore obtained

(i) By resolving the forces parallel to two axes at right angles and supposing them to act on the body as if all its mass were collected at the centre of gravity.

(ii) By taking moments about an axis through the C. of G. as if that axis were fixed in space.

EXAMPLE 1. A cylinder of radius r rolls down an inclined plane as in Fig. 199; to find the acceleration.

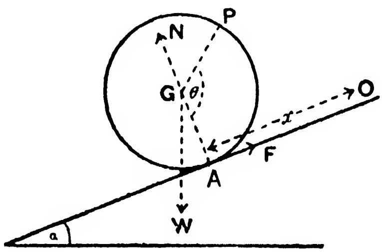

FIG. 199.

Suppose motion commenced when P was at O and that $OA=x$, then arc $AP=x=r\theta$. Let $AGP=\theta$, then angular acceleration of the cylinder about G is $\ddot{\theta}$. Let F be the frictional force at A (note that it does not equal μN as the point A does not slip).

If the external forces acted at G parallel to their original directions we should have:

$$\frac{W \sin \alpha - F}{W}=\frac{\ddot{x}}{g}. \quad \text{.........................(i)}$$

Rotation about an axis through G is caused by the force F;

$$\therefore \frac{Fr}{Wk^2}=\frac{\ddot{\theta}}{g}=\frac{\ddot{x}}{rg}, \quad \text{........................(ii)}$$

But $k^2 = \frac{r^2}{2}$; $\therefore$ from (i) and (ii) $W \sin \alpha - W\frac{\ddot{x}}{2g} = W\frac{\ddot{x}}{g}$;

$\therefore$ $\dot{x} = \frac{2}{3} g \sin \alpha$, which is $\frac{2}{3}$ the value if the surface were smooth.

By Energy Principle.

Let v be the velocity of the C. of G. when the cylinder has descended a distance x down the plane; then, since any point P is moving instantaneously round A with velocity

$$2v \cos \frac{(180 - \theta)}{2} = 2v \sin \frac{\theta}{2}$$ (see p. 323),

the angular velocity of P about A is

$$\frac{2v \sin \frac{\theta}{2}}{AP} = \frac{2v \sin \frac{\theta}{2}}{2r \sin \frac{\theta}{2}} = \frac{v}{r} = \omega;$$

$\therefore$ K.E. of the cylinder about an axis through A is $W(k^2 + r^2)\frac{\omega^2}{2g}$.

Since the cylinder does not slip at A, friction does no work;

$\therefore$ work done by gravity $= Wk^2 \frac{\omega^2}{2g} + Wr^2 \frac{\omega^2}{2g} = Wk^2 \frac{\omega^2}{2g} + W\frac{v^2}{2g}$.

EXAMPLE 2. A light thread is wound round a reel which is then allowed to fall, the end of the thread being held; find the acceleration of the reel.

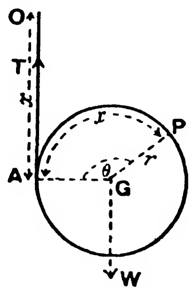

FIG. 200.

Suppose that initially P was at O and that OA $= x$, PGA $= \theta$, then $r\theta = x$ and $r\dot{\theta} = \ddot{x}$.

For the vertical motion

$$\frac{W-T}{W}=\frac{\ddot{x}}{g}. \qquad \text{(i)}$$

For the rotation

$$\frac{Tr}{\frac{Wr^2}{2}}=\frac{\ddot{\theta}}{g}=\frac{\ddot{x}}{rg}; \qquad \text{(ii)}$$

$$\therefore\ W-\frac{\ddot{x}}{rg}\frac{Wr}{2}=W\frac{\ddot{x}}{g};$$

by substituting for T in (i).

$$\therefore\ \ddot{x}=\frac{2g}{3}.$$

EXAMPLES XLIII.

1. A thin bar 4 ft. long weighing 10 lbs. swings from a horizontal position about an axis at one end. Find the force at the axis when the bar is vertical.

2. A sphere rolls down a plane without sliding; prove that its acceleration is $\frac{5}{7}$ of what it is when it slides down a smooth plane.

3. A solid uniform disc weighing 10 lbs. oscillates in a vertical plane through an angle of 120° about a horizontal axis through a point on its circumference. Find the pressure on the axis when the disc is in its lowest position.

4. A heavy wheel is placed between two parallel rails and rolls down with its axle on the rails. The swing radius of wheel and axle is k, and it is observed to travel s ft. in t seconds when the rails slope at α degrees. If the radius of the axle is r, prove that $g=\frac{2(k^2+r^2)s}{r^2t^2\sin\alpha}$.

5. If a cylinder of weight W and radius r rolls along a horizontal plane under the action of a force of P lbs. parallel to the plane, find the acceleration produced, and prove that

the rotating cylinder is equivalent to a body whose weight is $\frac{3}{2}W$ which is not rotating.

6. A carriage weighing W tons is mounted on wheels weighing W_1 tons, whose radius is r and radius of gyration k; find the acceleration with which the carriage will run down an inclined plane of angle α.

7. A uniform thin rod 4 ft. long weighing 6 lbs. can swing about an axis at one end. It is struck horizontally through the centre of oscillation so that it just makes a complete revolution; find the impulse of the blow.

8. A uniform flywheel is rolled along a rough horizontal plane by a constant force P applied horizontally to the axle by a string fastened on each side. Prove that the force of friction $= \frac{P}{3}$.

9. A barrel is rolled along a horizontal plane by a force of 12 lbs. applied tangentially at the top. If the barrel weighs 120 lbs., greatest diameter $2\frac{1}{2}$ ft., $k=1$ ft., find the acceleration and the magnitude of the frictional force.

10. A sphere rolls down between two parallel rods which are inclined at an angle α to the horizontal. If the rods are a ft. apart, and the radius of the sphere is r ft., find the acceleration of the sphere.

11. A wheel of 24″ diameter whose swing radius is 10″ rolls along the ground and meets a step 6″ high. Show that if the speed of the wheel is less than 6·16 ft./sec., it will not be able to lift itself on to the step. Assume the wheel inelastic and that there is no slipping at the point of contact with the square edge of the step. (C.U.)

12. A heavy sphere 1 ft. in diameter rolling along a level floor, strikes a rectangular step 2″ high at right angles to the step and jumps over it. Find the percentage change in angular velocity immediately after the blow.

13. A wheel of weight W, radius r, and radius of gyration k, is braked and skids along a road with velocity V; if the brake

is released, find the time taken by the wheel to acquire an angular velocity $\omega=\frac{V}{r}$ if W_1 is the load carried by the wheel, and μ the coefficient of friction.

14. A sphere of radius r is projected up an inclined plane of angle α with velocity V and is given an underhand spin ω; find the time that elapses before it turns back.

15. A uniform inelastic rod AB weighing 5 lbs. and of length 4 feet swings in a vertical plane about an axis at A. It is held in a horizontal position and released. Having turned through an angle of 30° it strikes against a rigid peg at a point 3 feet from A. Find the impulse produced on the axis and on the peg.

MISCELLANEOUS EXAMPLES.

V.

1. A door is 2 ft. wide and weighs 120 lbs. Find what impulsive torque would start it rotating about its hinges with an angular velocity of 2 rads./sec. If it comes to rest in 2 secs., find the frictional torque and the angle the door has turned through.

2. A cage weighing 1·5 tons is being raised up a mine shaft at a steady rate of 1 ft./sec. by a steel rope. The upper end of the rope is suddenly brought to rest, and the cage then oscillates up and down at the lower end of the chain, the free length of which is such that 5000 lbs. would stretch it 1 foot. Neglecting the inertia of the rope, find the period and amplitude of the oscillations and also the greatest additional pull in the rope.

3. A man of mass m is standing in a lift of mass M, which is descending with velocity V, the counterpoise being $M+m$. Suddenly the man jumps with an impetus which would raise him to a height h if he were jumping from the ground. Calculate the velocities of the man and the lift immediately after

the impulse, and find also their subsequent accelerations. Deduce that the height in the lift to which he jumps is

$$\frac{h(M+m)}{M+\frac{m}{2}}$$

4. AO is the vertical through O. A rod AB makes an angle θ with AO and a rod BC hinged to AB at B makes an angle $\phi(\phi>\theta)$ with AO. CO is perpendicular to AO. If the angular velocity of AB is ω and C is constrained to move along OC, find the instantaneous centre for the rod BC, and prove that the angular velocity of the rod BC is $\frac{BD}{BC}\omega$ if CB produced meets AO at D.

5. An armature shaft 2″ diameter, weighing 800 lbs., is placed so that it rests on two parallel knife edges sloping at $\frac{1}{8}$″ per foot, and it takes 10 secs. to roll 1 ft. Find its M.I., neglecting rolling friction. The armature is then put in its bearings and the machine run up to a speed of 600 revs./min. On switching off it is found that the speed falls to 540 revs./min. in 12 secs. Calculate the friction couple in lbs.-ft., assuming it constant.

W.

1. A small ring which can slide freely on a fixed vertical circle of radius a is connected with a particle of twice its mass by a string which passes over a pulley at one end of the horizontal diameter of the circle. If the ring be let fall from the other end of the diameter show that its velocity at the lowest point of the circle is $\sqrt{\{(5-2\sqrt{2})ag\}}$.

2. Two masses m_1, m_2 at A and B are connected by a weightless rod, and lie on a smooth horizontal table. The rod is struck at right angles to its length by a given blow of impulse I. Find the velocities of the masses, and show that the K.E. is least if the point of application of the blow divides AB in the ratio $m_2 : m_1$.

3. A smooth wedge weighing 5 lbs. can slide on a smooth horizontal plane. A weight of 1 lb. is placed on the sloping surface of the wedge 1 foot from the bottom edge, and allowed

to slide down. If the angle of the wedge is 30° and if the weight and wedge start from rest, prove that the weight reaches the bottom of the slope in about $\frac{1}{3}$ sec.

4. A projectile shot from the ground grazes the tops of two chimneys at heights of 36 ft. and 64 ft. which stand 96 ft. apart horizontally. If the time from chimney to chimney is 5 secs., prove that the initial velocity is 100 ft./sec., and show that its least possible value is 80 ft./sec.

5. A particle starts with velocity V and moves under a retardation equal to μ times the space described. Show that the distance travelled before coming to rest is $\frac{V}{\sqrt{\mu}}$.

X.

1. A train is drawn by an engine which exerts a constant pull at all speeds. The mass of the engine and train combined is 300 tons, and the resistance to motion varies as v^2. The maximum speed on the level is 60 mls./hr. and the H.P. then developed is 1500. Show that when climbing a slope of $\frac{1}{100}$ the maximum speed is 32 mls./hr. nearly.

2. A smooth inclined plane of mass M and inclination α rests on a smooth horizontal table. If a mass m is allowed to slide directly down the inclined plane, show that its acceleration relative to the plane is $\frac{M+m}{M+m\sin^2\alpha} \cdot g\sin\alpha$.

3. A dynamo armature weighing 2000 lbs. is carried on a vertical shaft which is supported on a flat footstep bearing 4″ in diameter. The radius of gyration of the armature is 1·5 ft. Assuming the pressure is uniformly distributed over the bearing area and that the mean coefficient of friction is 0·05, find how long the armature will take to come to rest from an initial speed of 450 revs. per min. under the influence of bearing friction only.

4. A sphere of radius c stands on a horizontal plane, and from a point in the plane at a distance $b<2c$ from the point of contact of the sphere, a particle is projected in a vertical

plane through the centre so as just to clear the sphere on both sides. Show that the velocity of projection is

$$\left[\frac{2b^2-2bc+c^2}{b-c}\cdot g\right],$$

but if $b>2c$ and the particle just clears the top of the sphere

$$v=\tfrac{1}{2}\left[\frac{g}{c}(b^2+16c^2)\right]^{\frac{1}{2}}.$$

5. A wheel whose M.I. $=40$ lb.-ft.^{2} is mounted so that it can at any instant be clutched to a concentric drum. The drum has a diameter of 1 ft. and its M.I. $=10$ lb.-ft.^{2}. A light rope wound round the drum is attached to a weight of 50 lbs. resting on the floor below. At what speed in revs./min. must the wheel be running so that when it is clutched to the drum the weight may be lifted through 10 ft. ?

Y.

1. A circle is rotated in its own plane about a point on its circumference with angular velocity ω, and a point P moves on the circle in the opposite direction with angular velocity 2ω relative to the circle ; prove that P moves in a straight line, and find its velocity.

2. Two fine cords are wrapped the same way round a solid circular cylinder of weight W, one at each end, and they are supported so as to hang vertically with the axis of the cylinder horizontal. The cylinder is released and unwraps itself from the cords as it descends. Find the tension in each cord and the vertical acceleration.

3. A circular hoop of radius a rolls with angular velocity ω along a rough horizontal table in a direction perpendicular to the edge. Prove that when it reaches the edge the centre of the hoop will fall a vertical distance $\dfrac{(g-a\omega^2)a}{2g}$ before the hoop leaves the edge. What happens if $a\omega^2>g$? (C.U.)

4. A box rests on the rack of a railway carriage. G is 10″ above and 3″ to the right of P, the point of contact of the box and the rail. Find the least acceleration of the rack in the

direction of the arrow which will cause the box to turn about the support at P. C U.)

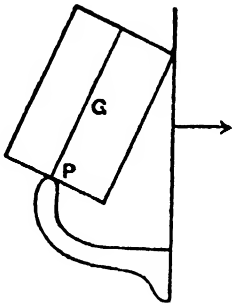

FIG. 201.

5. A motor bus has a wheel base a and the height of its C. of G. is h. If W_1 and W_2 are the loads on the back and front wheels respectively when at rest, $W_1 + \omega$, $W_2 - \omega$ the loads when the engine starts, μ the coefficient of friction, prove that the friction brought into play is $\frac{wa}{h}$ and that its maximum value is $\frac{\mu a W_1}{a - \mu h}$. Find the shortest time in which the bus can acquire a velocity of v ft./sec.

INVOLVING TRIGONOMETRIC AND LOGARITHMIC FUNCTIONS.

Z.

1. A particle slides down a smooth plane of inclination α and length l, and strikes a perfectly elastic horizontal plane. Find for what value of α the first bound will have a maximum range, and find the range.

2. A ship with engines stopped is gradually brought to rest by the resistance of the water. At one instant its speed is 10 ft./sec. and 1 min. later the speed is 6 ft./sec. For speeds below 2 ft./sec. the resistance may be taken to vary directly

as the speed and for speeds above this the resistance varies as v^2. Calculate how far the ship will drift from the place where the first observation was made before coming to rest.

3. A flywheel whose M.I. is 200 lbs.-ft.^{2} is rotating at the rate of 5 revolutions per sec., and its velocity is reduced in 10 secs. to 2 revs. per sec. by a fluid frictional resistance which produced a torque of $\lambda\omega$ lbs.-ft. where ω is the angular velocity of the flywheel. Find the initial value of this torque.

4. A valve consists of a horizontal metal plate kept on its seat by a spring. It is opened by the pressure of the edge of a circular disc of radius 3″ which revolves 4 times a sec. in a vertical plane about a point 2″ from its centre and 4″ vertically below the plate. Find how long the valve is open during each revolution of the disc and the rate at which it begins to open in ft./sec.

5. Show that the greatest height which a projectile with initial velocity V can reach on a vertical wall at a distance a from the point of projection is $\dfrac{V^2}{2g} - \dfrac{ga^2}{2V^2}$.

6. A motor car is driven by a constant force at all speeds, but the air resistance increases according to the square of the velocity; show that its acceleration on the flat is given by $\dfrac{dv}{dt} = k(v_0{}^2 - v^2)$ where v_0 is its full speed. Deduce that the time taken in getting up a speed v from rest is

$$\frac{1}{2kv_0}\log_e\left(\frac{v_0+v}{v_0-v}\right).$$

The mass of a motor car loaded is 5000 lbs., its H.P. is 50 and its full speed 75 mls./hr.; calculate the time required to get up a speed of 50 mls./hr. starting from rest. (C.S.C.)

7. The angular velocity of a rotating flywheel diminishes by 10 per cent. in 1 min. Find by how much per cent. it diminishes in the next minute if the couple resisting rotation varies as the angular velocity.

8. The couple which tends to right a sailing boat which has heeled over to an angle θ is $k \sin \theta$. A wind pressure producing

a torque which will maintain a steady inclination of 10° suddenly acts on the boat when vertical, and continues to act upon it. Show that the maximum angle to which the boat will heel is given by $\cos\theta + \theta \sin 10° - 1 = 0$.

9. The total wind and friction resistance on a motor car is of the form $a + bv^2$. A car which weighs 3200 lbs. requires 10 H.P. (effective) at 30 mls./hr. and 6 H.P. at $22\frac{1}{2}$ mls./hr. Find in what distance the speed will fall from 30 to 15 mls./hr. when coasting along a level road with clutch out.

10. The propulsive H.P. required to drive a steamer of 12,000 tons displacement at a steady speed of 20 knots is 15,000. Assuming the resistance proportional to v^2 and that the engines exert a constant propeller thrust at all speeds, find the initial acceleration when the steamer starts from rest. Find also how long it will take starting from rest to reach a speed of 10 knots. 1 knot = 100 ft./min.

ANSWERS.

CHAPTER XI.

Examples XXVI. (p. 184.)

1. $32t$; $50-32t$; 9. 192; -142; 9 ft./sec.
2. 32; 32; $10t+2$. 32; 32; 52 ft./sec^2.
3. $\theta=5$; $8t+6$. $\dot{\theta}=0$; 8. $\dot{\theta}=5$; 22. $\ddot{\theta}=0$; 8.
4. $\frac{k}{2}$. 5. $9\frac{3}{8}$. 6. $v=3(3-t)(1-t)$.
7. 8 mls./hr. 8. $\dot{v}=-2v^3$. 9. $\frac{3}{2}$ ft./sec.
10. 8 ft./sec. 11. 3·46 ft./sec. 12. 0·2 in./sec.
13. 2·8 mls./hr. 9·2 mls./hr. 2·43 p.m.

Examples XXVII. (p. 192.)

1. $y=\frac{4x}{5}-\frac{x^2}{625}$; $\frac{20-8t}{25}$; 101·3 ft./sec.; 9° 6′ below horizontal.
2. 45°; 45·25 ft./sec.; $\frac{dy}{dx}=1-t$; $t=1$; $x=32$, $y=16$; $x=13·54$, $y=10·68$.
3. $y=\frac{5}{3}x-\frac{7x^2}{150}$; 59° 2′. 4. $y=-\frac{11}{1200}x^2+\frac{43}{60}x+20$; 48°.
5. $\frac{u^2+v^2}{g}$. 6. $\frac{1}{3}$ H.P. 8. 3·2 ft.; 2·4 in.
9. $70y=350+135x-2x^2$; 51·4 ft./sec.; 62° 36′.
10. 2400 ft. 11. $\tan\theta=\frac{v}{u+U}$. 12. $400y=-x^2+60x+1600$.

Examples XXVIII. (p. 196.)

1. $x=2h-x\sin\beta$. 3. $a+\mathrm{OS}\cos 2\theta=h$.
4. $s\operatorname{cosec}\beta-2h \quad s$. 6. $\theta=90-\alpha$, $\tan\alpha=\frac{2h}{x}$.

Examples XXIX. (p. 204.)

1. 2 ft. from axle. **2.** 771 lbs. **3.** 7·9 tons.
4. Tang., $16\sqrt{3}$; normal, 32. $\frac{3}{2}$W.
5. 1·5 ft. from point of contact. **7.** $v = 1112$ ft./sec.
10. 7·85 tons. **13.** Leaves when $\omega^2 r = g$.

CHAPTER XII.

Examples XXX. (p. 214.)

1. $\frac{\omega^2}{2a}$. **2.** 6·4. **3.** 300. **4.** 57 H.P.
5. 2·7. **6.** 4 ft.-lbs. **7.** $\frac{(a+b-2l)(b-a)p}{2l}$.
8. $\frac{ka^2}{2}$ ft.-lbs. **9.** 31·4 ft.-lbs. **10.** 6290. **11** $\frac{E}{r}$.
12. 0·1 erg. **13.** 2381 ft.-tons. **14.** 9375 ft.-lbs.
15. 2812·5 cm.-grms. **16.** 104·5 ft.-tons. **17.** 7500 ft.-lbs.
18. $\frac{5280rm}{r+m}$; $\frac{m(2r-m)5280}{2r}$ ft.-tons.

Examples XXXI. (p. 223.)

1. 83·2 ft./sec.; 5408 ft.-lbs.; 309·3 ft. **2.** 1·52 secs.
3. 187·5 ft.-lbs.; 34·6 ft.-sec. **4.** 35 ft./sec., 816 ft.
6. $1·74 \times 10^8$ ft.-tons. **7.** 24,940 mls./hr.
8. 3·65 ft./sec.; 3·35 ft./sec. **9.** 297 H.P. **10.** 101·3 H.P.
11. 12 ft./sec. **14.** 5·06 ft./sec. **15.** 12·65 ft./sec.
16. $\frac{5}{2}$ Kg.-cm. **18.** 28·3 ft./sec.

CHAPTER XIII.

Examples XXXII. (p. 237.)

1. 1·15 ft. **2.** 25 lbs.-ft^2. **3.** $166\frac{2}{3}$ ft.-lbs.
4. 342·8 ft.-lbs. **5.** $\frac{5}{8}$ ft.-lbs. **6.** 29·6 ft.-lbs.
7. 11,106 ft.-lbs. **8.** 15·6 lbs.-ft^2. **9.** 1·89 ft.
10. 4 lbs.-ft^2. **11.** 565·5. **12.** W $\frac{3r^2}{2}$.
13. 1·48 ft. **14.** $168\frac{1}{3}$ lbs.-ft^2.

CHAPTER XIV.

Examples XXXIII. (p. 248.)

1. $26\frac{2}{3}$ rad./sec^2. **2.** $\frac{5}{8}$ lbs.-ft. **3.** 40 grams-cm.
4. 20 grams. **5.** 31·4 secs. **6.** $\frac{32}{63}$ lbs.
7. 2·4 rad./sec. **8.** 11·8 lbs.-ft.; 1·05 rad./sec^2.
9. 4 rad./sec^2.; 2·5 secs. **10.** 9·6 cm.-grams.

Examples XXXIV. (p. 254.)

1. 4·9 rad./sec. **2.** 4·7 lbs. **3.** 90·5 rad./sec.
4. 1·8 rad./sec. **5.** 5·7 rad./sec. **6.** 4·92 rad.
7. 14·6 ft./sec. **8.** 14·46 ft./sec.; 7·66 ft./sec.; 1·7 secs.
9. 13·93 rad./sec.; 6·4 ft.-lbs. **10.** 33·9 ft./sec.; 31 ft./sec.
11. 1·42 rad./sec. **12.** 0·415 lb.-ft. **13.** 2·75 rad./sec.
14. 5·8 ft./sec. **15.** 5·43 ft./sec. **16.** 9 lbs.; 9·6 lbs.
17. 66·2 ft. **18.** 88·8 lbs.

Miscellaneous Examples.

P. (p. 256.)

1. 63 ft./sec.; 12 ft./sec^2.; 2 secs. **3.** $T_1=300$, $T_2=600$ lbs.; yes.
4. $\tan\theta=\dfrac{\sqrt{a^2+b^2}+b}{a}$.

Q. (p. 257.)

1. $f\propto s^{\frac{1}{3}}$. **2.** $t=25$ secs.; $62\frac{1}{2}$ ft.
4. 41 secs.; $\frac{15}{896}$. **5.** 11·9 ft./sec.

R. (p. 258.)

1. 5·84 rad./sec. **2.** See Q. 3. **4.** By K.E., $T=\dfrac{fP}{g}\left[\dfrac{W+P}{W}\right]$.

S. (p. 259.)

1. Change of K.E. $=\frac{1}{2}$ ft.-ton; $T=2·83$ tons.
2. $\tan\alpha=\dfrac{u}{v+V}$. **3.** $T+t=a(s+l)+b(s+l)^2$, $V=\dfrac{1}{a+2bs+bl}$.
4. 3960. **5.** 813·3 ft.; $35\frac{3}{7}$ ft./sec.; 1693·3 ft.

T. (p. 260.)

1. $\theta\sqrt{\dfrac{12gk}{\mathrm{M}(l^2+a^2)}}$. 2. 717 sq. ft.; $4+\dfrac{x}{50}\cdot\dfrac{3}{2}$ ft.; 530,000 ft.-lbs.

3. 2·8 H.P. 4. $\mathrm{T}=\sqrt{\dfrac{2h}{g}}\left[\dfrac{2\mathrm{B}(\mathrm{A}+\mathrm{B})}{\mathrm{A}(\mathrm{A}-\mathrm{B})}+\dfrac{2\mathrm{B}}{\mathrm{A}}+1\right]$. 5. 1·38 secs.

U. (p. 261.)

1. 2M ft.-lbs. 3. $v=14e^{-2t}+16$; $s=22{\cdot}1$ ft.

4. 22·86 secs. 5. Work $=\mathrm{W}c\left(2+\dfrac{\pi}{2}\right)$, $e=56\%$.

7. $t=440\left(\dfrac{1}{v}-\dfrac{1}{u}\right)$. 8. $\mathrm{H}=r\sin\theta+\dfrac{\omega^2r^2\cos^2\theta}{2g}$.

9. $t_1=\dfrac{u^2\mathrm{M}}{\mathrm{R}g}$, $t_2=\dfrac{\mathrm{M}}{2\mathrm{R}g}(v^2-u^2)$, $s_1=\dfrac{1}{2}\dfrac{u^3\mathrm{M}}{\mathrm{R}g}$, $s_2=\dfrac{\mathrm{W}}{\mathrm{R}g}\left(\dfrac{v^3}{3}-\dfrac{u^3}{3}\right)$; 3′ 22″.

10. 27 ft.-cwts.

CHAPTER XV.

Examples XXXV. (p. 271.)

1. 1·15 ft./sec.; 132. 2. 0·64 ft./sec., 0·4 ft./sec. 3. 3·1.
4. 4·47 lbs. 5. 132. 7. 32·3 lbs.; 135·7 lbs.
8. 0·55 sec. 9. 1·54 lbs.-ft. 10. 1·6 ft./sec.
11. 5·06 ft.; 4·87 secs.; $\frac{29}{360}$. 12. 698 lbs.

Examples XXXVI. (p. 279.)

1. 99·4 cms. 2. 1·64 secs. 3. 1·77 secs.

4. $\ddot{x}+\dfrac{g}{d}x=0$. 5. 1·73; 1·58 secs.

6. $k_1^2=0{\cdot}32$, $t=1$ sec. nearly, $\omega=9$ rad./sec.

7. 2·49 ft.; $6{\cdot}64\times10^5$ ft.-lbs.

8. $\dfrac{[abc(c-a-b)]^{\frac{1}{2}}}{a+b}$. 9. $2\pi\sqrt{\dfrac{l\mathrm{I}}{gk}}$. 10. $\pi\sqrt{\dfrac{2l\mathrm{W}}{\mathrm{T}g}}$.

11. 106 grams; 694 cms./sec^2. 12. $2\pi\sqrt{\dfrac{l}{3g}}$.

CHAPTER XVI.

Examples XXXVII. (p. 288.)

1. 24·7 lbs. **2.** 2·4 lbs. **4.** 17·7 sec.-grams. Parallel to AB.
5. 55·2 lbs.; 23° 54′. **7.** 2·52 ft./sec.; 52° 24′.
10. 6·93 ft./sec. **11.** $w(v\cos\alpha - u) = Wu$.

12. $wV\cos\beta = Wu$, $\tan\alpha = \dfrac{V\sin\beta}{V\cos\beta + u}$.

13. $WV = wu$; $\tan\alpha = \dfrac{v}{u+V}$, Range $\dfrac{2uv}{g}$; $T_1 - T_2 = \dfrac{WV^2}{2lg} = \dfrac{Wh}{l}$.

15. $Fa = M\dfrac{v_1^2}{2g}$; $Mv_1 = (M+m)v_2$; $Fa = (M+m)\dfrac{v_3^2 - v_2^2}{2g}$.

16. $2a = \dfrac{V^2\cos^2\alpha}{g}$, $2a' = \dfrac{v^2\cos^2\beta}{g}$.

Examples XXXVIII. (p. 301.)

1. 160 lbs. **2.** 18° 22′. **3.** 53·2 ft./sec.; 18·2 ft./sec.
4. ·5 lb. **5.** 0·9 H.P.; 96·6%. **6.** 347.
7. 1·6 lbs.; 3·8 lbs.; 15·8 ft.-lbs. **8.** 623 lbs.; 1168 lbs.-ft.
9. 23° 48′; 36° 12′; 0·65 lb. **10.** $\dfrac{m(v\cos\theta + u\sin\theta)a}{g} = h_1$.

Examples XXXIX. (p. 308.)

1. 6 ft./sec.; $e = \frac{1}{4}$; $\frac{35}{64}$ ft.-lbs. **2.** $\frac{1}{4}$; 5 ft./sec.
3. 0·7. **4.** $\theta = 82°\ 24'$; $u = 6·1$ ft./sec. (10 lbs.), $v = 14·3$ ft./sec.

5. $E_1 = \dfrac{W_1W_2}{2g(W_1 + W_2)}(u_1 - u_2)^2$. **7.** $t_1 = \dfrac{2u\sin\beta}{g\cos\alpha}$, $t_2 = \dfrac{2eu\sin\beta}{g\cos\alpha}$.

8. $t = \dfrac{h}{v\sin\alpha}$. Horiz. vel. of B after impact $= v\cos\alpha$.

9. $\tan^{-1}\dfrac{\sin\theta}{e\cos\theta} + \theta = \dfrac{\pi}{2}$. **10.** $\dfrac{7\sqrt{3}}{19}$.

CHAPTER XVII.

Examples XL. (p. 316.)

1. 1·56 lbs. ft.-sec. **2.** 1·63. **3.** $\dfrac{I_1\omega_1 b^2}{a^2I_2 + b^2I_1}$; $\dfrac{I_1\omega_1 ab}{a^2I_2 + b^2I_1}$.

4. $\omega = 50·26$, 355 ft.-lbs.; 296 ft.-lbs. **6.** $\dfrac{I\omega}{g} = \dfrac{I\omega'}{g} + \dfrac{Mr^2\omega'}{g}$.

7. 147 lbs. 8. 4·6 ft./sec.; 27·6 rad./sec. 9. 3·6.

10. $\dfrac{W_2 v l}{W_2 l^2 + W_1(k^2+l^2)}$; $T = \dfrac{2\pi}{\sqrt{gl}} \dfrac{\sqrt{2gh}}{\omega}$. 11. 7·07 lbs.-ft.

12. 0·76 lb.-sec. 13. 2·4 rad./sec. 14. 7·7; 10·3. $\mu = 0·27$.

15. ·94; 2·95 ft./sec. 16. $W\omega \dfrac{5a^2}{g}$.

CHAPTER XVIII.

Examples XLI. (p. 330.)

1. 36·4 ft./sec.; 61·4 ft./sec².

2. $\dfrac{v}{u} = \dfrac{BO}{AO}$. 6. $\dfrac{\text{vel. } B}{\text{vel. } P} = \dfrac{BI}{PI} = \sin(\theta + \phi)$.

7. 14·14 ft./sec.; 7·55 ft./sec; 6·5 rad./sec.

8. $\omega = \dfrac{u}{AB \sin \theta}$. 9. $\dfrac{BN}{CB} . \omega$. 10. $\omega' KB = \omega BA$.

11. $\dfrac{1}{b}(a+b)\omega_3$; $\dfrac{1}{a}(a+b)\omega_3$. 12. $\dfrac{\text{vel. of } Q}{\text{vel. of } P} = \dfrac{5·7}{6·35} = 0·9$.

13. 6·4 rad./sec.; 6·4 rad./sec.; 45·6 lbs.-ft.

Examples XLII. (p. 339.)

1. $\dfrac{W_1 - W_2}{W_1 + W_2} . v$; $\left(\dfrac{W_1 - W_2}{W_1 + W_2}\right)^2 g$. 2. 2 ft./sec.

3. 7·03 ft./sec². 5. Zero. 1 ft. from 3 lbs.

7. 1·15 ft. 8. 2·93 ft.-lbs.

9. Both accelerations are const.; ∴ accel. of c. of g. is const. Locus is str. line joining mid points of the two paths.

10. Zero. 11. 3·5 ft.-lbs. 12. 12 ft./sec.; g.

Examples XLIII. (p. 347.)

1. 25 lbs. 3. $16\frac{2}{3}$ lbs. 5. $\dfrac{2Pg}{3W}$.

6. $\dfrac{(W+W_1)gr^2 \sin \alpha}{W_1 k^2 + (W+W_1)r^2}$. 7. 2·6. 9. 3·9 ft./sec².; 2·6 lbs.

10. $\dfrac{5(4r^2 - a^2)g \sin \alpha}{28r^2 - 5a^2}$. 11. $W\dfrac{V}{g} . \dfrac{1}{2} + W\left(\dfrac{5}{6}\right)^2 \dfrac{V}{g} = W \dfrac{61}{36} \dfrac{\omega'}{g}$.

12. 24 %. 13. $\dfrac{Wk^2\omega}{gr\mu W_1}$.

14. $\dfrac{rV - k^2\omega}{rg \sin \alpha}$. 15. 0·12; 0·96.

Miscellaneous Examples.

V. (p. 349.)

1. **5; 2 rads.**

2. 0·9 sec. ; 2900 lbs.

3. $\sqrt{2gh}-V$; $\dfrac{m\sqrt{2gh}}{2M+m}+V$; accel. of lift, $\dfrac{mg}{2M+m}$.

5. $I=87$; $C=1\cdot4$.

W. (p. 350.)

2. $Ig\left(\dfrac{1}{M}+\dfrac{c}{m_1 l}\right)$; $Ig\left(\dfrac{1}{M}-\dfrac{c}{m_2 l}\right)$, where c is the distance from G.

3. Accel. of wedge $=\dfrac{W\sin 30\cos 30 g}{5+\sin^2 30}$; of body $=\frac{1}{4}g$.

X. (p. 351.)

3. 10 mins. (nearly).

5. 337·5 revs /min.

Y. (p. 352.)

1. $2a\omega\sin\omega t$.

2. $T=\dfrac{W}{6}$, $a=\dfrac{2g}{3}$.

3. $W2a^2\left(\dfrac{\omega_1^2-\omega^2}{2g}\right)-Wh$, $\cos\theta=\dfrac{a\omega_1^2}{g}$.

4. 9·6 ft./sec²

Z. (p. 353.)

1. $\tan^2\alpha=2$, $s=\dfrac{8l}{3\sqrt{3}}$.

2. 2348·5 ft.

3. $\lambda=0\cdot573$, $C=18$.

4. $\frac{1}{12}$ sec ; 2·4 ft./sec.

6. 55·3 secs.

7. 10 %.

8. $\theta k\sin 10=\int k\sin\theta\, d\theta$.

9. 712 ft.

10. $a=\frac{99}{112}$, 61·6 secs.

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